Adiabatic limits of the Seiberg-Witten equations on Seifert manifolds

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Introduction

The "old" instanton theory naturally lead to the instanton Floer homology of a 3-manifold N as the missing piece in a general gluing formula for the Donaldson invariants. Similarly, the Seiberg-Witten theory leads to the "monopole homology" which is the Floer homology of the Seiberg-Witten functional defined by a spin^c manifold (see [KM], [Mar], [MW], [Wa]).

The first main difficulty in understanding this homology comes from the fact that its defining chains are not as explicit as the chains which generate the instanton homology. In the latter case these are the flat connections on an $SU(2)$ bundle which are well understood both topologically and geometrically. The meaning of the chains in monopole homology is far from obvious and the only explicit computations were made when the 3-manifold N is a product $S^1 \times \Sigma$ where Σ is a surface of genus ≥ 2 (see [D] or [MST]). The equations are tractable in this case is because $S^1 \times N$ is a *Kähler manifold*. As was pointed out in [D], the solutions of the 3D Seiberg-Witten equations coincide with the S^1-invariant solutions of the 4D Seiberg Witten equations. Fortunately, on a Kähler manifold the solutions of these equations can be described explicitly.

If now N is the total space of a principal S^1 bundle of *nonzero* degree over a surface Σ then $S^1 \times N$ admits a natural complex structure but this time the manifold cannot be Kähler for the simple reason that the first Betti number is odd.

We analyze the Seiberg-Witten equations on a special class of 3-manifolds, namely those which admit a Killing vector field of constant pointwise length and satisfy an additional technical condition . Topologically, these manifolds must be Seifert fibered manifolds.

On such manifolds the Dirac operators have an especially nice form and in particular the Seiberg-Witten equations can be further dissected. We are interested in the behavior of the solutions of the Seiberg-Witten equations as the metric degenerates

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in the direction of the Killing vector field. This corresponds to collapsing the fibers of the Seifert fibration.

The paper is divided into four parts. The first part studies in detail the differential geometry of the metric almost contact (m.a.c.) manifolds. In particular, we distinguish a special class of such manifolds the so called *Killing m.a.c* manifolds. These are Riemann manifolds which admit a Killing vector field ζ of pointwise length 1. Any oriented Killing m.a.c 3-manifold is diffeomorphic to a Seifert manifold and moreover any Seifert 3-manifold admits a Killing m.a.c structure. A special class of Killing m.a.c. 3-manifolds consists of the (K, λ) manifolds, $\lambda \in \mathbb{R}$. They are characterized by the condition

$$d\eta(x) \equiv 2\lambda * \eta$$

where η denotes the 1-form dual to the Killing vector field ζ and λ is a constant. These manifolds are also characterized by the fact that their product with S^1 admits a natural integrable complex structure.

The total space of a principal S^1-bundle admits a natural (K, λ)-structure described for the first time by Boothby and Wang. We present a 1-parameter family of such nice metric structures and we explicitly compute its differential geometric invariants: the Levi-Civita connection, the Ricci and the scalar curvature. Factoring with suitable groups of isometries one can construct many other interesting examples. In particular, in subsection §1.4 we show that any Seifert manifold admits a natural (K, λ)-structure. More precisely, the Thurston geometries on Seifert manifolds (from the list of 6 described in [S]) are (K, λ) structures. The scalar λ is proportional with the Euler number of the Seifert fibration should be regarded as a measure of "twisting " of the fibration. As the metric degenerates (and so the fibers become shorter and shorter) λ will go to zero and thus the fibration will become "less and less twisted".

As was observed by several authors ([ENS], [V]) the (K, λ) structures with $\lambda > 0$ are links of quasihomogeneous singularities. Their Thurston geometry is (almost) uniquely determined by the analytical structure of the singularity and conversely, (see [Ne] or [Sch]) the Thurston geometry fixes the analytical type of the singularity.

The second part is devoted to Dirac operators on m.a.c manifolds. The spinor bundles corresponding to the various spin^c structures can be very nicely described in the almost-contact language. We introduce two different classes of Dirac operators and compare them. Also we establish a commutator identity which is an important ingredient in the study of adiabatic limits. In the third part we introduce the 3-dimensional Seiberg-Witten equations and study the adiabatic limits of solutions as the metric degenerates in the direction of ζ. On an S^1- bundle of degree ℓ over a surface of genus g the "adiabatic picture" has some similarities with the exact descriptions in [D], [MST] or [Mun] when the 3-manifold is the trivial S^1-bundle over a surface. However new phenomena occur when $|\ell| \geq 2g > 0$. In this case, for a certain range of spin^c structures the adiabatic limit consists only of the genuine reducible solutions of the Seiberg-Witten equations which are adiabatic invariants. We conclude this section by showing that if the fibers are sufficiently small then the reducible

solutions are the only solutions of the Seiberg-Witten equations (corresponding to spin^c-structures in the above range). They determine a collection of tori of dimension $2g$ which is nondegenerate in a Morse theoretic sense.

The last section of the paper illustrates how one can use the adiabatic knowledge to actually produce irreducible solutions of the Seiberg-Witten equations. More precisely we show that the isolated, irreducible adiabatic solutions can always be perturbed to genuine solutions of the Seiberg-Witten equations corresponding to metrics with $\delta \gg 1$. The idea is that these adiabatic solutions almost solve the Seiberg-Witten equations and if all goes well the technique of [T] can be used to detect nearby genuine irreducible solutions. The problem is technically complex since all the important analytical quantities such as eigenvalues, Sobolev constants etc. vary with δ and one must understand quite accurately the manner in which this happens. There are fortunately many magical coincidences which make this endeavour less painful than expected.

Contents

1 Metric almost contact structures

The tangent bundle of any oriented Riemann 3-manifold is trivial. In particular, its structural group $SO(3)$ can be reduced to $SO(2) \cong U(1)$. Such a reduction is called an almost contact structure. In this section we will discuss the special differential geometric features of a 3-manifold equipped with an almost contact structure. Most of these facts are known (see [B], [YK]) but we chose to present them in some detail in order to emphasize the special 3-dimensional features.

§1.1 Basic objects Consider an oriented 3-manifold N.

Definition 1.1 (a) An *almost contact structure* on N is a nowhere vanishing 1-form $\eta \in \Omega^1(N)$.
(b) A Riemann metric g on N is said to be compatible with an almost contact structure η if $|\eta(x)|_g = 1$ for all $x \in N$. A *metric almost contact structure* (m.a.c) on N is a pair (η, g)= (almost contact structure, compatible metric).

Consider a m.a.c structure (η, g) on the oriented 3-manifold N. A local, oriented, orthonormal frame $\{\zeta_0, \zeta_1, \zeta_2\}$ of TN is said to be *adapted to the m.a.c. structure* if ζ_0 is the metric dual of η. The dual coframe of an adapted frame $\{\zeta_0, \zeta_1, \zeta_2\}$ has the form $\{\eta^0, \eta^1, \eta^2\}$ where $\eta^0 = \eta$ and $*\eta = \eta^2 \wedge \eta^2$. In the sequel we will operate exclusively with adapted frames.

Denote by $Cl(N)$ the bundle of Clifford algebras generated by T^*N equipped with the induced metric. The quantization map

$$\text{exterior algebra} \rightarrow \text{Clifford algebra}$$

(see [BGV]) induces a map

$$\mathfrak{q} : \Lambda^* T^* N \rightarrow Cl(N).$$

On the other hand $\Lambda^* T^* N$ has a natural structure of $Cl(N)$-module so that via the quantization map we can construct an action of $\Lambda T^* M$ on itself

$$\mathfrak{c} : \Lambda^* T^* N \rightarrow \mathrm{End}\,(\Lambda^* T^* N)$$

called *Clifford multiplication.*

On a m.a.c. 3-manifold (N, η, g) the Clifford multiplication by $*\eta$ has a remarkable property. More precisely

$$\mathfrak{c}(*\eta)\langle\eta\rangle^\perp = \langle\eta\rangle^\perp \subset \Lambda^* T^* N.$$

If (η^0, η^1, η^2) is a local coframe then the bundle $\langle\eta\rangle^\perp$ is locally spanned by η^1, η^2 and $\mathfrak{c}(*\eta)$ acts according to the prescription

$$\mathfrak{c}(*\eta) : \eta^1 \mapsto \eta^2 \ , \ \eta^2 \mapsto -\eta^1.$$

In particular, we notice that both $\mathfrak{c}(*\eta)$ and $-\mathfrak{c}(*\eta)$ define complex structures on the real 2-plane bundle $\langle\eta\rangle^\perp$.

Definition 1.2 The complex line bundle $(\langle\eta\rangle^{\perp}, -\mathfrak{c}(*\eta))$ is called the *canonical line bundle* of the m.a.c. 3-manifold (N, η, g) and is denoted by $\mathcal{K} = \mathcal{K}_{\eta,g}$.

When viewed as a real bundle $\mathcal{K}$ (and hence $\mathcal{K}^{-1}$ as well) has a natural orientation. We have an isomorphism of *oriented* real vector bundles

$$T^*N \cong \langle -\eta\rangle \oplus \mathcal{K} \cong \langle\eta\rangle \oplus \mathcal{K}^{-1} \tag{1.1}$$

where $\langle -\eta\rangle$ (resp $\langle\eta\rangle$)denotes the real line bundle spanned and oriented by $-\eta$ (resp η).

§1.2 The structural equations of a m.a.c. manifold Consider an oriented m.a.c. 3 manifold (N, η, g) and denote by ∇ the Levi-Civita connection of the metric g Fix and adapted local frame of $\{\zeta_0, \zeta_1, \zeta_2\}$ and denote by $\{\eta^0, \eta^1, \eta^2\}$ the dual coframe. The connection 1-form of ∇ with respect to these trivializations can be computed using Cartan's structural equation. More precisely if

$$d\begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \end{bmatrix} = \begin{bmatrix} 0 & -A & B \\ A & 0 & -C \\ -B & C & 0 \end{bmatrix} \wedge \begin{bmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \end{bmatrix} \tag{1.2}$$

(A, B, C are real valued 1-forms locally defined on N) then

$$\begin{cases} \nabla\zeta_0 = & & -A\otimes\zeta_1 & +B\otimes\zeta_2 \\ \nabla\zeta_1 = & A\otimes\zeta_0 & & -C\otimes\zeta_2 \\ \nabla\zeta_2 = & -B\otimes\zeta_0 & +C\otimes\zeta_1 & \end{cases} \tag{1.3}$$

We will analyze the above equations when (N, g) is a *Killing m.a.c.* manifold i.e. the vector ζ is Killing. For $j = 0, 1, 2$ we set

$$A_j = i_{\zeta_j}A, \quad B_j = i_{\zeta_j}B, \quad C_j = i_{\zeta_j}C.$$

Since $L_\zeta g = 0$ we have the equality

$$g(\nabla_X\zeta, Y) = -g(X, \nabla_Y\zeta) \quad \forall X, Y \in \mathrm{Vect}\,(N).$$

We substitute X and Y with pairs of basic vectors ζ_i, ζ_j and we obtain the following identities.

$$A_0 = B_0 = A_1 = B_2 = 0 \tag{1.4}$$

and

$$A_2(x) = B_1(x). \tag{1.5}$$

Set $\lambda(x) := A_2(x) = B_1(x)$. The structural equations now yield

$$d\eta = 2\lambda(x) * \eta.$$

Thus the scalar $\lambda(x)$ is independent of the local frame used i.e. it is an *invariant* of the Killing m.a.c. structure (N, η, g).

Differentiating the above equality we deduce $0 = 2d(\lambda(x) * \eta)$ so that

$$\partial_\zeta \lambda(x) = 0. \tag{1.6}$$

Note that

$$\nabla_\zeta \zeta_1 = -C_0(x)\zeta_2, \quad \nabla_\zeta \zeta_2 = C_0(x)\zeta_1.$$

Thus $C_0(x)$ defines the infinitesimal rotation of $\langle \zeta \rangle^\perp$ produced by the parallel transport along ζ. Hence this is another invariant of the Killing m.a.c. structure and will be denoted by $\varphi(x)$. Finally set

$$b(x) = \lambda(x) + \varphi(x).$$

Note for further references that

$$[\zeta_1, \zeta_0] = \nabla_{\zeta_1}\zeta_0 - \nabla_0 \zeta_1 = b(x)\zeta_2 \tag{1.7}$$

and

$$[\zeta_2, \zeta_0] = \nabla_{\zeta_2}\zeta_0 - \nabla_{\zeta_0}\zeta_2 = -b(x)\zeta_1. \tag{1.8}$$

For each $\delta > 0$ denote by g_δ the anisotropic deformation of g defined by

$$g_\delta(X, X) = g(X, X) \ \text{ if } \ g(X, \zeta) \equiv 0.$$

$$g_\delta(\zeta, \zeta) = \frac{1}{\delta^2}.$$

Set

$$\eta_\delta = \eta/\delta.$$

Note that $d\eta_\delta = 2\lambda\delta^{-1} *_\delta \eta_\delta$. Set $\lambda_\delta = \frac{\lambda}{\delta}$. Note that

$$b_\delta = \delta b \quad \varphi_\delta = \delta b - \lambda/\delta = \delta\varphi + \left(\delta - \frac{1}{\delta}\right)\lambda. \tag{1.9}$$

Anisotropic deformations as above were discussed also in [YK] where they were named D-homotheties.

Remark 1.3 (a) If λ is not constant and c is a regular value of λ then the level set $\lambda^{-1}(c)$ is a smooth embedded surface in N and ζ is a nowhere vanishing tangent vector field along $\lambda^{-1}(c)$. If N is compact this implies $\lambda^{-1}(c)$ is an embedded torus. The trajectories of ζ (which wander around the level sets $\lambda^{-1}(c)$) define an 1-dimensional Riemannian foliation on N. Such foliations on compact 3-manifolds where completely classified in [Ca]. In particular, the topological type of N is severely restricted (see also Proposition 1.4 below).

(b) The above computations show that a Killing m.a.c structure defines a normal almost contact structure on N. This means that the almost complex structure J on $S^1 \times N$ defined by

$$J : \frac{\partial}{\partial \theta} \mapsto \zeta \quad \text{and} \quad \zeta_1 \mapsto \zeta_2$$

is integrable. See [B] or [YK] for more details.

We can now easily compute the sectional curvatures $\langle R(\zeta_j, \zeta)\zeta, \zeta_j\rangle$. More precisely we have

$$\begin{aligned} R(\zeta_1, \zeta)\zeta &= \left(\nabla_{\zeta_1}\nabla_{\zeta} - \nabla_{\zeta}\nabla_{\zeta_1} - \nabla_{[\zeta_1,\zeta]}\right)\zeta \\ &= -\nabla_{\zeta}(\lambda(x)\zeta_2) - b(x)\nabla_{\zeta_2}\zeta = \lambda^2(x)\zeta_1. \end{aligned}$$

Hence

$$\langle R(\zeta_1, \zeta)\zeta, \zeta_1\rangle = \lambda^2(x) \tag{1.10}$$

and similarly

$$\langle R(\zeta_2, \zeta)\zeta, \zeta_2\rangle = \lambda^2(x). \tag{1.11}$$

Hence the scalar curvature of N is determined by

$$s = 2\kappa + 4\lambda^2(x) \tag{1.12}$$

where $\kappa(x)$ denotes the sectional curvature of the plane spanned by ζ_1 and ζ_2.

The connection ∇ defines via orthogonal projections a connection $\nabla^{\perp}$ on the complex line bundle $\operatorname{Ann}\eta = \langle\zeta\rangle^{\perp}$. (The complex structure is defined by $\mathbf{i}\zeta_1 = \zeta_2$. More precisely

$$\left\{ \begin{array}{rcrr} \nabla^{\perp}\zeta_1 & = & & -C(x)\otimes\zeta_2 \\ \nabla^{\perp}\zeta_2 & = & C(x)\otimes\zeta_1 & \end{array} \right.$$

where $C(x) = \varphi(x)\eta + C_1(x)\eta_1 + C_2(x)\eta_2$. Using the complex structure in $\langle\zeta\rangle^{\perp}$ we can locally describe $\nabla^{\perp}$ as

$$\nabla^{\perp} = d - \mathbf{i}C(x).$$

Under anisotropic deformations this connection 1-form changes to

$$C_\delta = \varphi_\delta\eta_\delta + C_1\eta_2 + C_2\eta_2 = \frac{1}{\delta}\varphi_\delta\eta + C_1\eta_2 + C_2\eta_2.$$

The equality (1.9) shows that as $\delta \to \infty$ the form C_δ converges to

$$C_\infty := b(x)\eta + C_1(x)\eta_1 + C_2(x)\eta_2 = \lambda(x)\eta + C(x).$$

If we denote by ∇^{∞} the limiting connection we have

$$\nabla^{\infty} = \nabla^{\perp} - \mathbf{i}\lambda(x)\eta. \tag{1.13}$$

Denote by $F^\perp$ the curvature of the connection $\nabla^\perp$ and by σ the scalar

$$\sigma = \langle F^\perp(\zeta_1, \zeta_2)\zeta_2, \zeta_1\rangle. \tag{1.14}$$

It is not difficult to show that σ is independent of the local frame and so is an invariant of the Killing m.a.c. structure. With respect to this frame it has the description

$$\sigma(x) = \partial_{\zeta_1} C_2 - \partial_{\zeta_2} C_1 - (C_1)^2 - (C_2)^2.$$

Note that the anisotropic deformation introduced in §1.1 does not change $\nabla^\perp$ so that

$$\sigma_\delta(x) = \sigma(x).$$

Using the structural equations of ∇ we deduce

$$\nabla_{\zeta_1}\nabla_{\zeta_2}\zeta_2 = (\partial_{\zeta_1} C_2)\zeta_1 - C_1 C_2 \zeta_2.$$

$$\nabla_{\zeta_2}\nabla_{\zeta_1}\zeta_2 = (\lambda C_1 - \partial_{\zeta_2}\lambda)\zeta + \left(\lambda^2(x)\zeta_1 + \partial_{\zeta_2} C_1\right)\zeta_1 - C_1 C_2 \zeta_2$$
$$[\zeta_1, \zeta_2] = -2\lambda(x)\zeta + C_1\zeta_1 + C_2\zeta_2.$$

Hence

$$\begin{aligned} \kappa(x) &= \langle R(\zeta_1, \zeta_2)\zeta_2, \zeta_1\rangle \\ &= \partial_{\zeta_1} C_2 - \partial_{\zeta_2} C_1 - (C_1)^2 - (C_2)^2 - \lambda^2(x) + 2\lambda(x)\varphi(x) \\ &= \sigma(x) - \lambda^2(x) + 2\lambda(x)\varphi(x). \end{aligned} \tag{1.15}$$

In particular, using (1.12) we deduce

$$s(x) = 2\{\sigma(x) + \lambda^2(x) + 2\lambda(x)\varphi(x)\}.$$

Note that

$$s_\delta(x) = 2\{\sigma(x) + \lambda^2/\delta^2 + 2\lambda(b - \lambda/\delta)/\delta\}$$

so that

$$\lim_{\delta\to\infty} s_\delta(x) = 2\sigma(x). \tag{1.16}$$

Thus for δ very large $\sigma(x)$ is a good approximation for the scalar curvature.

Proposition 1.4 Assume N is a compact oriented Killing m.a.c 3-manifold. Then N is diffeomorphic to an oriented Seifert fibered 3-manifold. Conversely, any compact oriented Seifert 3-manifold admits a Killing m.a.c structure.

Proof The Killing m.a.c. structures (η, g) with respect to the *fixed* metric g are parameterized by the unit sphere in the Lie algebra of compact Lie group of isometries $\mathrm{Isom}\,(N, g)$. In particular, the group $\mathrm{Isom}\,(N, g)$ has positive dimension. If this is the case the maximal torus containing ζ induces at least one fixed-point-free S^1 action

on N (slight perturbations of ζ in the Lie algebra of $\mathrm{Isom}(N, g)$ will not introduce zeroes of the corresponding vector field on N). Hence N must be a Seifert manifold.

Conversely, given a Seifert fibered manifold N denote by ζ the generator of the fixed-point-free S^1 action and for each $\theta \in S^1$ denote by $\mathcal{R}_\theta$ its action on N Define $\mathfrak{M}_\zeta$ as the collection of Riemann metrics g on N such that $|\zeta(x)|_g \equiv 1$. Note that $\mathfrak{M}$ is convex and

$$\mathcal{R}_\theta^* \mathfrak{M}_\zeta \subset \mathfrak{M}_\zeta.$$

The S^1-average of any $g \in \mathfrak{M}_\zeta$ is ζ -invariant and thus defines a Killing m.a.c structure on N. $\square$

When the invariant $\lambda(x)$ is constant we will call the structure (N, η, g) a (K, λ)-manifold. The $(K, 1)$-manifolds are also known as K-*contact* manifolds (cf. [B], [YK]). In dimension 3 this notion coincides with the notion of Sasakian manifold.

On a (K, λ) manifold the sectional curvature of any plane containing ζ is λ^2 and the full curvature tensor is completely determined by the the sectional curvature κ of the planes spanned by ζ_1 and ζ_2. The scalar curvature of N is

$$s = 2\kappa + 4\lambda^2.$$

Using the structural equations we deduce that with respect to the adapted frame $\{\zeta_0, \zeta_1, \zeta_2\}$ the Ricci curvature has the form

$$\mathrm{Ric} = \begin{bmatrix} 2\lambda^2 & 0 & 0 \\ 0 & \kappa + \lambda^2 & -\kappa \\ 0 & -\kappa & \kappa + \lambda^2 \end{bmatrix}$$

For any oriented (K, λ) 3-manifold (not necessarily compact) we denote by $\mathfrak{K}_N$ the group of isomorphism of the (K, λ) structure i.e. orientation preserving isometries which invariate η. For any discrete subgroup $\Gamma \subset \mathfrak{K}_N$ acting freely and discontinuously on N we obtain a covering

$$N \to N/\Gamma.$$

Clearly N/Γ admits a natural (K, λ) structure induced from N. In §1.4 we will use this simple observation to construct (K, λ)-structures on any Seifert 3-manifold.

§1.3 The Boothby-Wang construction In this subsection we describe some natural (K, λ) structures on the total space of a principal S^1-bundle over a compact oriented surface. Except some minor modifications this construction is due to Boothby and Wang, [BW] (see also [B]).

Consider $\ell \in \mathbb{Z}$ and denote by N_ℓ the total space of a degree ℓ principal S^1 bundle over a compact oriented surface of genus g: $S^1 \hookrightarrow N_\ell \xrightarrow{\pi} \Sigma$. We orient N_ℓ using the rule

$$\det TN_\ell = \det TS^1 \wedge \det T\Sigma.$$

Assume Σ is equipped with a Riemann metric h_b such that $\mathrm{vol}_{h_b}(\Sigma) = \pi$. Recall that if $\omega \in \mathbf{i}\mathbb{R} \otimes \Omega^1(N_\ell)$ is a connection form on N_ℓ then $d\omega$ descends to a 2-form Ω on Σ: the curvature of ω. Moreover

$$\int_\Sigma c_1(\Omega) = \frac{\mathbf{i}}{2\pi}\int_\Sigma \Omega = \ell = \int_\Sigma \frac{\ell}{\pi} dv_{h_b}.$$

Notice in particular that $\mathbf{i}\Omega$ is cohomologous to $2\ell dv_{h_b}$. Now pick a connection form ω such that

$$\mathbf{i}\Omega = 2\ell dv_{h_b}.$$

Denote by ζ the unique vertical vector field on N_ℓ such that $\omega(\zeta) \cong \mathbf{i}$ and set $\eta_\delta = -\mathbf{i}\omega/\delta$. Thus $\eta_\delta(\zeta) \cong 1/\delta$. Notice that $\mathrm{Ann}\,\eta$ coincides with the horizontal distribution H_ω defined by the connection ω. Now define a metric h on N_ℓ according to the prescriptions.

$$h(\zeta, \zeta) = 1/\delta^2$$

and

$$h(X, Y) = h_b(\pi_* X, \pi_* Y) \;\text{ if } X,\ Y \text{ are horizontal.}$$

Clearly $L_\zeta h =$ i.e. ζ is a Killing vector field. Moreover

$$\delta d\eta_\delta = -\Omega = -2\ell\pi^* dv_{h_b} = -2\ell * \eta_\delta.$$

In other words, we have constructed a $(K, -\ell/\delta)$ structure on N_ℓ.

We conclude this subsection by describing the geometry of N_ℓ in terms of the geometry of Σ. The only thing we need to determine is the curvature κ introduced in the previous subsection in terms of the sectional curvature Σ. It is not difficult to see this coincides with the invariant $\sigma(x)$ defined in (1.14). Because of the special geometry of this situation formula (1.15) can be further simplified.

To achieve this we will use again the structural equations. Denote by $\{\psi^1, \psi^2\}$ a local, oriented orthonormal coframe on Σ and set $\eta^j = \pi^*\psi^j$, $j = 1, 2$. Then $\{\eta^0 = \eta, \eta^1, \eta^2\}$ is an adapted coframe on N_ℓ.

The structural equations of Σ have the form

$$d\begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} \wedge \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} \tag{1.17}$$

where θ is a 1-form locally defined on Σ. Set

$$\theta_j = i_{\pi_*\zeta_j}\theta, \quad j = 1, 2$$

Since $d\eta^j = \pi^* d\psi^j$ we deduce that $d\eta^j$ is horizontal. On the other, hand using (1.2) we deduce

$$d\eta^1 = A \wedge \eta^0 - C \wedge \eta^2 = (A_2 + C_0)\eta^2 \wedge \eta^0 - C_1\eta^1 \wedge \eta^2.$$

This implies $C_0 = -A_2 = \ell/\delta$ so that

$$b(x) \equiv 0 \tag{1.18}$$

and

$$\varphi \equiv \ell/\delta. \tag{1.19}$$

Using (1.17) we deduce $d\psi^1 = -\theta \wedge \psi^2 = -\theta_1 \psi^1 \wedge \psi^2$ which yields

$$C_1 = \theta_1. \tag{1.20}$$

Similarly one shows

$$C_2 = \theta_2. \tag{1.21}$$

Using the above two equalities and (1.13) we conclude that the limiting connection ∇^∞ on $\mathcal{K}$ is none-other than the pullback by $\pi : N \to \Sigma$ of the Levi-Civita connection on the canonical line bundle $K_\Sigma \to \Sigma$ i.e.

$$\nabla^\infty = \pi^* \nabla^K. \tag{1.22}$$

Using (1.15) and (1.19) we deduce

$$\kappa = \sigma - 3\ell^2/\delta^2 \tag{1.23}$$

and in particular the scalar curvature of N_ℓ is given by

$$s_N = 2(\sigma - \ell^2/\delta^2). \tag{1.24}$$

Remark 1.5 Let N denote the total space of a principal S^1 bundle over an oriented surface Σ, *not necessarily compact*. If ω denotes a connection form on N such that $-\mathbf{i}d\omega$ descends to a *constant* multiple of the volume form on Σ then the previous computations extend verbatim to this case and one sees that in this situation one also obtains a (K, λ)-structure on N.

For example let N denote the unit tangent bundle of the hyperbolic plane $\mathbb{H}^2$. The Levi-Civita connection on $\mathbb{H}^2$ induces an S^1-connection ω on N. Then

$$-\mathbf{i}d\omega = -1 d\, vol_{\mathbb{H}^2}$$

since $\mathbb{H}^2$ has constant curvature $\equiv -1$. Thus N has a natural $(K, 1)$ (= Sasakian) structure.

The group of isometries of $\mathbb{H}^2$ is $PSL(2, \mathbb{R})$ and induces an action on N which preserves the above Sasakian structure. (In fact, N is isomorphic with $PSL(2, \mathbb{R})$ and via this isomorphism the above action is precisely the usual left action of a Lie group on itself.) If now $\Gamma \subset PSL(2, \mathbb{R})$ is a Fuchsian group with a compact fundamental domain then N/Γ is a compact Seifert manifold with a natural Sasakian structure.

§1.4 **Geometric Seifert structures** The main result of this subsection shows that any compact, oriented, Seifert 3-manifold admits a (K, λ) structure. This will follow easily from the description of the *geometric Seifert structures* in [JN] or [S].

Theorem 1.6 Any compact, oriented, Seifert 3-manifold admits a (K, λ) structure.

Proof We will begin by reviewing the basic facts about the geometric Seifert structures in a form suitable to the application we have in mind. For details we refer to [JN] or [S] and the references therein.

A geometric structure on a manifold M is a complete locally homogeneous Riemann metric of finite volume. The universal cover of a manifold M equipped with a geometric structure is a homogeneous space which we will call the *model* of the structure. It is known that if a 3-manifold admits a geometric structure then its model belongs to a list of 8 homogeneous spaces (see[S]).

Any Seifert manifold admits a geometric structure corresponding to one of the following 6 models:

$$S^2 \times \mathbb{E}^1, \quad \mathbb{E}^3, \quad \mathbb{H}^2 \times \mathbb{E}^1, \quad S^3, \quad N, \quad P\tilde{S}L$$

where $\mathbb{E}^k$ denotes the k-dimensional Euclidean space, $\mathbb{H}^2$ denotes the hyperbolic plane, N denotes the Heisenberg group equipped with a left invariant metric and $P\tilde{S}L$ denotes the universal cover of $PSL(2, \mathbb{R}) \cong \mathrm{Isom}\,(\mathbb{H}^2)$.

According to [RV], the Seifert manifolds which admit N as a model are the non-trivial S^1 bundles over a torus and as we have seen in the previous subsection such manifolds admit (K, λ) structures.

The Seifert manifolds which admit geometric structures by $\mathbb{E}^3$ are flat space form and are completely described in [Wo]. One can verify directly that these admit natural (K, λ) structures.

S^3 has a natural $(K, 1)$ structure as the total space of the Hopf fibration $S^3 \to S^2$. Any Seifert manifold modeled by S^3 is obtained as the quotient by a finite group of *fiber preserving isometries*. Thus they all inherit a $(K, 1)$ structure.

If $\mathbb{X}$ is a model other than S^3 or $\mathbb{E}^3$ then the group of isomorphisms which fix a given point $x \in \mathbb{X}$ fixes a tangent direction at that point. So $\mathbb{X}$ has an $\mathrm{Aut}\,(\mathbb{X})$-invariant tangent line field. This line field fibers $\mathbb{X}$ over $\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$.

For $\mathbb{X} = \mathbb{S}^2 \times \mathbb{E}^1, \mathbb{H}^2 \times \mathbb{E}^1$ this is the obvious fibration. $P\tilde{S}L$ can be alternatively identified with the universal cover of the unit tangent bundle $T_1\mathbb{H}^2$ of $\mathbb{H}^2$. It thus has a natural line fibration which coincides with the fibration abstractly described above. Note that the $(K, 1)$ structure on $T_1\mathbb{H}_2$ constructed at the end of §1.3 lifts to the universal cover $P\tilde{S}L$.

If $\mathbb{X}$ is one of of these remaining three models denote by $\mathrm{Aut}_f(X) \subset \mathrm{Aut}\,(X)$ the subgroup preserving the above line fibration (as an oriented fibration). Note that each of them admits a (K, λ) structure and Aut_f is in fact a group of isomorphisms of this structure.

The trivial Seifert manifold $S^2 \times S^1$ presents a few "pathologies" as far as geometric Seifert structures are concerned (see [JN]) but we do not need to worry since it obviously admits a $(K, 0)$ structure.

The other Seifert manifolds which admit geometric structures modeled by $S^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$ or $P\tilde{S}L$ can be obtained as quotients $\Pi \backslash \mathbb{X}$ where Π is some subgroup of Aut_f. Thus Π invariates the universal (K, λ) structures on these models and therefore the quotients will admit such structures as well.

The list of Seifert manifolds is complete. Note in particular that the Seifert manifolds geometrized by N, $\mathbb{H}^2 \times \mathbb{E}^1$ and $P\tilde{S}L$ are $K(\pi, 1)$'s and hence they cannot admit metrics of nonnegative scalar curvature. $\square$

Remark 1.7 The above analysis can be refined to offer an answer to the question raised in [We]: which Seifert manifolds admit Sasakian structures. The answer is simple. A Seifert manifold admits a Sasakian structure if and only if its (rational) Euler class is negative. According to [NR], these are precisely the Seifert manifolds which can occur as links of a quasi-homogeneous singularity. This extends (in the $3D$ case) the previous result of [Sas] concerning Sasakian structures on Brieskorn manifolds.

This fact was observed by many other authors (see [ENS], [Ne], [V]). In fact this geometry of the link is in most cases a complete invariant of the analytic type of the singularity (see [Ne], [Sch]).

2 Dirac operators on 3-manifolds

In this section we discuss the relationships between *spin*^{c} structures and m.a.c. structures on a 3-manifold. On any m.a.c. 3-manifold, besides spin^c Dirac operators there exists another natural Dirac operator which imitates the Hodge-Dolbeault operator on a complex manifold. We will analyze the relationships between them.

§2.1 3-dimensional spinorial algebra We include here a brief survey of the basic facts about the representations of $Spin(3) \cong SU(2)$. Denote by Cl_3 the Clifford algebra generated by $V = \mathbb{R}^3$ and by $\mathbb{H}$ the skew-field of quaternions. Consider an orthogonal basis $\{e_0, e_1, e_2\}$ of V.

It is convenient to identify $\mathbb{H}$ with $\mathbb{C}^2$ via the correspondence

$$\mathbb{H} \ni q = u + \mathbf{j}v \mapsto \left[\begin{array}{c} u \\ v \end{array}\right] \in \mathbb{C}^2 \tag{2.1}$$

where

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \quad u = a + b\mathbf{i}, \quad v = c - d\mathbf{i}.$$

For each quaternion q denote by L_q (resp. R_q) the left (resp. the right) multiplication by q. $R_{\mathbf{i}}$ defines a complex structure on $\mathbb{H}$ and the correspondence (2.1) defines an isomorphism of complex vector spaces.

The Clifford algebra Cl_3 can be represented on $\mathbb{C}^2 \cong \mathbb{H}$ using the correspondences

$$e_0 \mapsto L_{\mathbf{i}} \longleftrightarrow \mathbf{c}(e_0) = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}. \tag{2.2}$$

$$e_1 \mapsto L_{\mathbf{j}} \longleftrightarrow \mathbf{c}(e_1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.3}$$

$$e_2 \mapsto L_{\mathbf{k}} \longleftrightarrow \mathbf{c}(e_2) = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}. \tag{2.4}$$

The restriction of the above representation to $Spin(3) \subset Cl_3$ defines the complex spinor representation of $Spin(3)$

$$\mathbf{c} : Spin(3) \to \mathrm{Aut}\,(\mathbb{S}_3).$$

The Clifford multiplication map

$$\mathbf{c} : \Lambda^* V \stackrel{\mathfrak{q}}{\to} Cl_3 \to \mathrm{End}\,(\mathbb{S}_3) \tag{2.5}$$

identifies $\Lambda^1 V$ with the space of traceless, skew-hermitian endomorphisms of $\mathbb{S}_3$. It extends by complex linearity to a map from $\Lambda^1 V \otimes \mathbb{C}$ to the space of traceless endomorphisms of $\mathbb{S}_3$. In particular, the purely imaginary 1-forms are mapped to selfadjoint endomorphisms.

For each $\phi \in \mathbb{S}_3$ denote by $\tau(\phi)$ the endomorphism of $\mathbb{S}_3$ defined by

$$\tau(\phi)\psi = \langle \psi, \phi \rangle \phi - \frac{1}{2}|\phi|^2 \psi.$$

The map τ plays a central role in the 3-dimensional Seiberg-Witten equations.

If we identify $\mathbb{S}_3$ with $\mathbb{C}^2$ as above and if ϕ has the form

$$\phi = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

the $\tau(\phi)$ has the description

$$\tau(\phi) = \frac{1}{2}\sum_{i=0}^{2} \langle \phi, \mathbf{c}(e_i)\phi \rangle \mathbf{c}(e_i) = \begin{bmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\overline{\beta} \\ \overline{\alpha}\beta & \frac{1}{2}(|\beta|^2 - |\alpha|^2) \end{bmatrix}. \tag{2.6}$$

It is not difficult to check that the nonlinear map

$$\tau : \mathbb{S}_3 :\to \mathrm{End}\,(\mathbb{S}_3)$$

is $Spin(3)$ equivariant.

We want to describe some of the invariant-theoretic features of the structure:

(oriented Euclidean 3 dimensional space + distinguished unit vector).

This is the algebraic counterpart of a m.a.c. structure on an oriented 3-manifold.

Assume V is an oriented Euclidean space which has a distinguished unit vector, say e_0. The group of isomorphisms of this structure is $U(1) \cong S^1 \cong SO(2)$. The group $Spin(3)$ acts naturally on V. The Lie algebra of the subgroup H of $Spin(3)$ which fixes e_0 is generated by $e_1e_2 = \mathbf{q}(*e_0)$ and can be identified with $\underline{u}(1)$ via the correspondence

$$e_1e_2 \mapsto \mathbf{i} \in \underline{u}(1).$$

This tautologically identifies H with S^1. The representation of H on $\mathbb{S}_3$ is no longer irreducible and consequently $\mathbb{S}_3$ splits as a direct sum of irreducible H-modules. Alternatively, this splitting can be described as the unitary spectral decomposition of $\mathbb{S}_3$ defined by the action of e_1e_2 on $\mathbb{S}_3$. According to (2.3) and (2.4) we have

$$\mathbb{S}_3 \cong \mathbb{S}_3(\mathbf{i}) \oplus \mathbb{S}_3(-\mathbf{i}).$$

The action of H on $\mathbb{S}_3(\mathbf{i})$ is the tautological S^1 representation, while the action on $\mathbb{S}_3(-\mathbf{i})$ is the conjugate of the tautological representation.

§2.2 3-dimensional spin geometry Consider a compact, oriented, m.a.c. 3-manifold (N, η, g). Since $w_2(N) = 0$ the manifold N is spin. To understand the relationship between the spin structures on N and the m.a.c structure we need to consider gluing data of TN compatible with the m.a.c. structure.

Consider a good, open cover $\{U_\alpha\}$ of N and a gluing cocycle

$$g_{\alpha\beta} : U_{\alpha\beta} \to SO(2) \cong U(1)$$

defining TN. The cocycle is valued in $SO(2)$ since TN has a distinguished section ζ, the dual of η. Note that $g_{\alpha\beta}$ defines a complex structure in the real 2-plane bundle $\langle\zeta\rangle^\perp$. It is not difficult to see that $\langle\zeta\rangle^\perp \cong \mathcal{K}^{-1}$ *as complex line bundles.*

A spin structure on TN is a lift of this cocycle to an H-valued cocycle, where H is the subgroup of $Spin(3)$ defined at the end of §2.1. Using the tautological identification $H \cong S^1$ we can identify the cover $H \to SO(2)$ with

$$S^1 \xrightarrow{z^2} S^1.$$

Hence a spin structure is defined by a cocycle

$$\tilde{g}_{\alpha\beta} : U(1) \cong SO(2) \to U(1)$$

such that $\tilde{g}_{\alpha\beta}^2 = g_{\alpha\beta}$. In other words, $\tilde{g}_{\alpha\beta}$ defines a square root of $\mathcal{K}^{-1}$. Moreover, two such lifts define isomorphic square roots if and only if they are cohomologous so there

exists a bijective correspondence between the square roots of $\mathcal{K}^{-1}$ (or, equivalently $\mathcal{K}$) and the spin structures on N.

Now, fix a spin structure on N defined by a lift

$$\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to H. \tag{2.7}$$

The complex spinor bundle $\mathbb{S}$ of this spin structure is associated to the principal H-bundle defined by (2.7) via the representation

$$H \hookrightarrow Spin(3) \to \mathrm{Aut}\,(\mathbb{S}).$$

As we have already seen this splits as $\tau_1 \oplus \tau_{-1}$ where τ_1 denotes the tautological representation of $S^1 \cong H$ and τ_{-1} is its conjugate.

The component τ_1 defines the square root of of $\langle\zeta\rangle^\perp$ i.e. the line bundle $\mathcal{K}^{-1/2}$ characterizing the chosen spin structure. We have thus shown that a choice of a m.a.c. structure on N produces a splitting

$$\mathbb{S} \cong \mathcal{K}^{-1/2} \oplus \mathcal{K}^{1/2}. \tag{2.8}$$

Once we have fixed a spin structure it is very easy to classify the spin^c structures: they are bijectively parameterized by the complex line bundles $L \to N$. The complex spinor bundle associated to the spin^c structure defined by the line bundle L is

$$\mathbb{S}_L \cong \mathbb{S} \otimes L \cong \mathcal{K}^{-1/2} \otimes L \oplus \mathcal{K}^{1/2} \otimes L$$

An important important special case is when $L = \mathcal{K}^{-1/2}$. In this case

$$\mathbb{S}_{\mathcal{K}^{-1/2}} \cong \mathcal{K}^{-1} \oplus \underline{\mathbb{C}} \stackrel{def}{=} \mathbb{S}_\eta$$

where we denoted by $\underline{\mathbb{C}}$ the trivial complex line bundle over N.

Remark 2.1 Our sign conventions differ from those of [MST]. There they chose $\mathbb{S}(\mathbf{i}) \cong \mathcal{K}^{1/2}$. The overall effect is a permutation of rows and columns in the block description of the geometric Dirac operator of §2.4.

§2.3 Pseudo Dolbeault operators The complex bundle $\mathbb{S}_\eta$ introduced in §2.2 was *a priori* defined in terms of a fixed spin structure on N but *a posteriori*, the spin structure becomes irrelevant. This is similar to complex manifolds where $\Lambda^{0,*}T^*$ is a complex spinor bundle of the spin^c structure canonically associated to the complex manifold. In that case the Dolbeault operator is a Dirac operator compatible with the Clifford structure. Moreover it is a geometric Dirac operator if the manifold is Kähler.

In this subsection want to pursue this analogy a little further. In the process we will construct an operator which behaves very much like the Dolbeault operator in the complex case.

Denote by $\mathcal{P}$ the real 2-plane bundle $\langle\eta\rangle^{\perp}$. We orient $\mathcal{P}$ using the complex structure $-\mathfrak{c}(*\eta)$ which identifies it with $\mathcal{K}$. Consider now a complex hermitian vector bundle $E \to N$. Any connection ∇ on E defines an operator

$$\nabla : C^{\infty}(E) \to C^{\infty}(T^*N \otimes E).$$

Now observe that

$$T^*N \otimes E \cong (\langle\eta\rangle \oplus \mathcal{P}) \otimes E \cong E \,\oplus\, (\mathcal{K} \otimes E) \,\oplus\, (\mathcal{K}^{-1} \otimes E).$$

Hence for any section $\psi \in C^{\infty}(E)$ the covariant derivative ψ orthogonally splits into three components:

$$\nabla_{\zeta}\psi \in C^{\infty}(E)$$

$$^{\flat}\nabla\psi \in C^{\infty}(\mathcal{K} \otimes E)$$

and

$$^{\flat}\overline{\nabla}\psi \in C^{\infty}(\mathcal{K}^{-1} \otimes E).$$

It terms of a local adapted frame $\{\zeta_0 = \zeta, \zeta_1, \zeta_2\}$ we have

$$^{\flat}\nabla\psi = \varepsilon \otimes (\nabla_1 - \mathbf{i}\nabla_2)\psi$$

$$^{\flat}\overline{\nabla}\psi = \bar{\varepsilon} \otimes (\nabla_1 + \mathbf{i}\nabla_2)$$

where

$$\nabla_j = \nabla_{\zeta_j},\ \ \varepsilon = \frac{1}{\sqrt{2}}(\eta^1 + \mathbf{i}\eta^2),\ \ \bar{\varepsilon} = \frac{1}{\sqrt{2}}(\eta^1 - \mathbf{i}\eta^2).$$

For example when $E = \underline{\mathbb{C}}$ then $^{\flat}\overline{\nabla}\psi \in C^{\infty}(\mathcal{K}^{-1})$. In this case if ∇ is the trivial connection d we will write $\partial_{\flat}$ (resp. $\bar{\partial}_{\flat}$) instead of $^{\flat}\nabla$ (resp. $^{\flat}\overline{\nabla}$). Notice that if $E = \mathcal{K}^{-1}$ then $^{\flat}\nabla\psi \in C^{\infty}(\underline{\mathbb{C}})$.

Coupling a metric connection ∇^E on E with the Levi-Civita connection $\nabla^{\perp}$ on $\mathcal{K}^{\pm}$ we obtain connections $\nabla^{E,\pm}$ on $E \otimes \mathcal{K}^{\pm}$ and in particular operators

$$^{\flat}\nabla^{E,-} : C^{\infty}(E \otimes \mathcal{K}^{-}) \to C^{\infty}(E),\ \ ^{\flat}\overline{\nabla}^{E,+} : C^{\infty}(E \otimes \mathcal{K}) \to C^{\infty}(E).$$

Using the above explicit form of the operators $^{\flat}\nabla$ and $^{\flat}\overline{\nabla}$ and the structural equations of the background m.a.c structure we obtain the following result.

Lemma 2.2 For any hermitian vector bundle E and any hermitian connection ∇^E on E we have

$$^{\flat}\nabla^* = {}^{\flat}\overline{\nabla}^{E,+},\ \ ^{\flat}\overline{\nabla}^* = {}^{\flat}\nabla^{E,-}$$

where the upper $*$ denotes the formal adjoint.

The Levi-Civita connection on (N,g) induces via orthogonal projection a connection $\nabla^{\perp}$ on $\mathcal{K}^{-1}$ compatible with the hermitian structure. The *pseudo Dolbeault* operator on (N,η,g) is the first order partial differential operator $\mathfrak{d} : C^{\infty}(\mathbb{S}_{\eta}) \to C^{\infty}(\mathbb{S}_{\eta})$ which in terms of the splitting $\mathbb{S}_{\eta} \cong \mathcal{K}^{-1} \otimes \underline{\mathbb{C}}$ has the block decomposition

$$\mathfrak{d}_N = \left[\begin{array}{cc} \mathbf{i}\nabla^{\perp}_{\zeta} & \bar{\partial}_{\flat} \\ & \\ {}^{\flat}\nabla^{\perp} & -\mathbf{i}\partial_{\zeta} \end{array}\right]$$

where ${}^{\flat}\nabla^{\perp}$ is obtained as above starting from the connection $\nabla^{\perp}$ on $\mathcal{K}^{-1}$. More generally, consider the complex spinor bundle $\mathbb{S}_L$ associated to the spin^c structure defined by the complex line bundle L

$$\mathbb{S}_L \cong \mathcal{K}^{-1/2} \otimes L \oplus \mathcal{K}^{1/2} \otimes L.$$

Using the connection ∇ on $\mathcal{K}$ and a connection A on L we obtain connections $\nabla^{\pm}$ on $\mathcal{K}^{\pm 1/2} \otimes L$. We can produce a twisted pseudo-Dolbeault operator $\mathfrak{d}_L = \mathfrak{d}_{L,A}$ on $\mathbb{S}_L$ described by the block decomposition

$$\mathfrak{d}_L = \mathfrak{d}_{L,A} = \left[\begin{array}{cc} \mathbf{i}\nabla^{-}_{\zeta} & {}^{\flat}\overline{\nabla}^{+} \\ & \\ {}^{\flat}\nabla^{-} & -\mathbf{i}\nabla^{-}_{\zeta} \end{array}\right].$$

Lemma 2.2 shows that $\mathfrak{d}_L$ is formally selfadjoint. Note that

$$\mathfrak{d}_N = \mathfrak{d}_{\mathcal{K}^{-1/2}}.$$

When L is trivial we set $\mathfrak{d}_L = \mathfrak{d}_0$.

Denote by $\mathbf{F}^{\pm}$ the curvature of $\nabla^{\pm}$. We want to analyze the "commutator"

$$[{}^{\flat}\overline{\nabla}^{+}, \nabla^{+}_{\zeta}] \stackrel{def}{=} {}^{\flat}\overline{\nabla}^{+} \circ \nabla^{+}_{\zeta} - \nabla^{-}_{\zeta} \circ {}^{\flat}\overline{\nabla}^{+}.$$

assuming N is a Killing m.a.c manifold. Choose a local adapted frame $\{\zeta, \zeta_1, \zeta_2\}$ with dual coframe $\{\eta, \eta_1, \eta_2\}$. For any $\psi \in C^{\infty}(\mathcal{K}^{1/2} \otimes L)$ we have

$$[{}^{\flat}\overline{\nabla}^{+}, \nabla^{+}_{\zeta}]\psi = \bar{\varepsilon} \otimes (\nabla^{+}_1 + \mathbf{i}\nabla^{+}_2)\nabla^{+}_{\zeta}\psi - \nabla^{-}_{\zeta}\{\bar{\varepsilon} \otimes (\nabla^{+}_1 + \mathbf{i}\nabla^{+}_2)\}\psi$$

$$= \bar{\varepsilon} \otimes \left(\nabla^{+}_1 + \mathbf{i}\nabla^{+}_2\right)\nabla^{+}_{\zeta}\psi - (\nabla^{\perp}\bar{\varepsilon}) \otimes \left(\nabla^{+}_1 + \mathbf{i}\nabla^{+}_2\right)\psi$$

$$-\bar{\varepsilon} \otimes \left(\nabla^{+}_{\zeta}(\nabla^{+}_1 + \mathbf{i}\nabla^{+}_2)\right)\psi$$

$$= \bar{\varepsilon} \otimes \left\{(\nabla^{+}_1 + \mathbf{i}\nabla^{+}_2)\nabla^{+}_{\zeta} - \nabla^{+}_{\zeta}(\nabla^{+}_1 + \mathbf{i}\nabla^{+}_2)\right\}\psi - (\nabla^{\perp}_{\zeta}\bar{\varepsilon}) \otimes (\nabla^{+}_1 + \nabla^{+}_2)\psi$$

(use $\nabla^{\perp}_{\zeta}\bar{\varepsilon} = -\mathbf{i}\varphi(x)\bar{\varepsilon}$)

$$= \bar{\varepsilon} \otimes (\mathbf{F}^{+}(\zeta_1,\zeta) + \mathbf{i}\mathbf{F}^{+}(\zeta_2,\zeta) + \nabla^{+}_{[\zeta_1,\zeta]} + \mathbf{i}\nabla^{+}_{[\zeta_2,\zeta]})\psi + \mathbf{i}\varphi(x)\bar{\varepsilon} \otimes (\nabla^{+}_1 + \mathbf{i}\nabla^{+}_2)\psi$$

(use (1.7) and (1.8))

$$= \bar{\varepsilon} \otimes (\mathbf{F}^+(\zeta_1,\zeta) + \mathbf{i}\mathbf{F}^+(\zeta_2,\zeta) + b(x)\nabla_2^+ - \mathbf{i}b(x)\nabla_1^+)\psi + \mathbf{i}\varphi(x)\bar{\varepsilon}\otimes(\nabla_1^+ + \mathbf{i}\nabla_2^+)\psi$$

$$= \bar{\varepsilon} \otimes \Big\{\mathbf{F}^+(\zeta_1,\zeta) + \mathbf{i}\mathbf{F}^+(\zeta_2,\zeta) - \mathbf{i}b(x)(\nabla_1^+ + \mathbf{i}\nabla_2^+)\Big\}\psi + \mathbf{i}\varphi(x)\bar{\varepsilon}\otimes(\nabla_1^+ + \mathbf{i}\nabla_2^+)\psi$$

$$= \bar{\varepsilon} \otimes (\mathbf{F}^+(\zeta_1,\zeta) + \mathbf{i}\mathbf{F}^+(\zeta_2,\zeta))\psi - \mathbf{i}\lambda\bar{\varepsilon}\otimes(\nabla_1^+ + \mathbf{i}\nabla_2^+)\psi.$$

We have thus proved

$$[{}^\flat\overline{\nabla}^+, \nabla_\zeta^+] = -\mathbf{i}\lambda(x)\,{}^\flat\overline{\nabla}^+ + \bar{\varepsilon}\otimes\Big\{\mathbf{F}^+(\zeta_1,\zeta) + \mathbf{i}\mathbf{F}^+(\zeta_2,\zeta)\Big\} \tag{2.9}$$

When dealing with the Seiberg-Witten equations it is convenient to describe the curvature term in the above formula in terms of the curvature F_A of L. We will use the formula

$$\mathbf{F}^+ = F_{\mathcal{K}^{1/2}\otimes L} = F_A + \frac{1}{2}F_{\mathcal{K}}.$$

Hence we need to explicitly describe the curvature of $\mathcal{K}$ equipped with the connection $\nabla^\perp$.

Note that $\mathcal{K}^{-1}$ can be identified with the bundle $\langle\zeta\rangle^\perp$ equipped with the complex structure

$$\mathbf{i}\zeta_1 = \zeta_2 \quad \mathbf{i}\zeta_2 = -\zeta_1.$$

We will compute the curvature of this line bundle using the structural equations of $\nabla^\perp$

$$\nabla^\perp\zeta_j = -\mathbf{i}C\otimes\zeta_j \;\; j=1,2.$$

Assuming N is a Killing m.a.c manifold we deduce after some simple manipulations

$$F_{\mathcal{K}^{-1}}(\zeta_1,\zeta) = -\mathbf{i}(-\partial_1\varphi(x) + \partial_\zeta C_1 + b(x)C_2)$$

$$F_{\mathcal{K}^{-1}}(\zeta_2,\zeta) = -\mathbf{i}(-\partial_2\varphi(x) + \partial_\zeta C_2 - b(x)C_1).$$

Set $\mu(x) = C_1 + \mathbf{i}C_2$. Some elementary algebra shows

$$F_{\mathcal{K}^{-1}}(\zeta_1,\zeta) + \mathbf{i}F_{\mathcal{K}^{-1}}(\zeta_2,\zeta) = \mathbf{i}(\partial_1 + \mathbf{i}\partial_2)\varphi - (\mathbf{i}\partial_\zeta + b(x))\mu.$$

Hence

$$\bar{\varepsilon}\otimes(F_{\mathcal{K}}(\zeta_1,\zeta) + \mathbf{i}F_{\mathcal{K}}(\zeta_2,\zeta) = -\mathbf{i}\bar{\partial}_\flat\varphi + \bar{\varepsilon}\otimes\{(\mathbf{i}\partial_\zeta + b(x))\mu\}. \tag{2.10}$$

We now want to clarify the "mysterious" term $(\mathbf{i}\partial_\zeta + b(x))\mu$ in the above formula. To achieve this we will use the structural equations (1.2). Thus

$$d\eta^1 = A\wedge\eta - C\wedge\eta^2 = -b(x)\eta\wedge\eta^2 - C_1 * \eta$$

and

$$d\eta^2 = -B\wedge\eta + C\wedge\eta^1 = b(x)\eta\wedge\eta^1 - C_2 * \eta.$$

Temporarily set $\omega = \sqrt{2}\varepsilon = \eta^1 + \mathbf{i}\eta^2$. The above equalities yield

$$d\omega = \mathbf{i}b(x)\eta \wedge \omega - \mu * \eta. \tag{2.11}$$

Differentiating the last equality we deduce

$$d(\mu * \eta) = \mathbf{i}d(b(x)\eta \wedge \omega)$$

i.e.

$$(\partial_\zeta \mu) dvol_g \;=\; \mathbf{i}\{db(x) \wedge \eta \wedge \omega + b(x)d\eta \wedge \omega - b(x)\eta \wedge d\omega\}.$$

The middle term in the right-hand-side of the above formula cancels. The third term can be computed using (2.11). Hence

$$(\partial_\zeta \mu) dvol_g = \mathbf{i}\{db(x) \wedge \eta \wedge \omega + b(x)\eta \wedge *\eta\}$$

or equivalently,

$$(\partial_\zeta - \mathbf{i}b(x))\mu \, dvol_g \;=\; \mathbf{i}db(x) \wedge \eta \wedge \omega.$$

Since

$$db(x) \wedge \eta \wedge \omega = (-\mathbf{i}\partial_1 b(x) + \partial_2 b(x)) dvol_g$$

we deduce

$$(\mathbf{i}\partial_\zeta + b(x))\mu = \mathbf{i}(\partial_1 + \mathbf{i}\partial_2)b(x).$$

In a more invariant form

$$\bar{\varepsilon} \otimes (\mathbf{i}\partial_\zeta + b(x))\mu = \mathbf{i}\bar{\partial}_\flat b(x). \tag{2.12}$$

Using the above equality in (2.10) we deduce

$$\bar{\varepsilon} \otimes (F_{\mathcal{K}}(\zeta_1, \zeta) + \mathbf{i}F_{\mathcal{K}}(\zeta_2, \zeta)\} = -\mathbf{i}\bar{\partial}_\flat \varphi + \mathbf{i}\bar{\partial}_\flat b(x) = \mathbf{i}\bar{\partial}_\flat \lambda(x). \tag{2.13}$$

Set

$$F_A^{0,1} := \bar{\varepsilon} \otimes F_A(\zeta, \zeta_1 + \mathbf{i}\zeta_2)$$

and

$$F_A^{1,0} := \varepsilon \otimes F_A(\zeta, \zeta_1 - \mathbf{i}\zeta_2).$$

We can now rephrase the commutativity relation (2.9) as

$$[{}^\flat\overline{\nabla}^+, \nabla_\zeta^+] = -\mathbf{i}\lambda(x)\, {}^\flat\overline{\nabla}^+ + \frac{\mathbf{i}}{2}\bar{\partial}_\flat \lambda(x) - F_A^{0,1} \tag{2.14}$$

By passing to formal adjoints we deduce using Lemma 2.2

$$[{}^\flat\nabla^-, \nabla_\zeta^-] = [({}^\flat\overline{\nabla}^+)^*, \nabla_\zeta^-] := ({}^\flat\overline{\nabla}^+)^* \nabla_\zeta^- - \nabla_\zeta^+ ({}^\flat\overline{\nabla}^+)^* = \mathbf{i}\lambda({}^\flat\overline{\nabla}^+)^* - \frac{\mathbf{i}}{2}({}^\flat\bar{\partial})^*\lambda - F_A^{1,0}. \tag{2.15}$$

On a (K, λ) manifold the commutativity relations (2.14) and (2.14) further simplify. Fix a spin^c structure determined by a line bundle L and choose a connection A on L. It will be extremely convenient to introduce two new differential operators $Z_A, T_A : C^\infty(\mathbb{S}_L) \to C^\infty(\mathbb{S}_L)$ defined by the block decompositions

$$Z_A = \left[\begin{array}{cc} \mathbf{i}\nabla_\zeta^{A,-} & 0 \\ 0 & -\mathbf{i}\nabla_\zeta^{A,+} \end{array} \right]$$

$$T_A = \left[\begin{array}{cc} 0 & {}^\flat\overline{\nabla}^+ \\ ({}^\flat\overline{\nabla}^+)^* & 0 \end{array} \right]$$

Then the commutativity relations (2.14) and (2.15) can be simultaneously rephrased as an anti-commutator identity

$$\{Z_A, T_A\} := Z_A T_A + T_A Z_A = -\lambda T_A - \mathbf{i} \left[\begin{array}{cc} 0 & -F_A^{0,1} \\ F_A^{1,0} & 0 \end{array} \right]. \tag{2.16}$$

We want to emphasize here that the operators Z_A, T_A and $F_A^{0,1}$ *depend on the background m.a.c. structure*. Both Z_A and $F_A^{0,1}$ *will be affected by anisotropic deformations* while T_A is invariant.

§2.4 Geometric Dirac operators In this subsection we will analyze the geometric Dirac operators on a 3-manifolds and in particular we will relate them with the pseudo-Dolbeault operators of §2.3.

Consider an oriented Killing m.a.c manifold (N, η, g) with a *fixed* spin structure. Denote by $\mathbb{S} \cong \mathcal{K}^{-1/2} \oplus \mathcal{K}^{1/2}$ the bundle of complex spinors associated to this structure.

We begin by first recalling the construction of the canonical connection on $\mathcal{K}$. Pick a local adapted frame $\{\zeta_0 = \zeta, \zeta_1, \zeta_2\}$ and denote by σ_j the Clifford multiplication by ζ_j, $j = 0, 1, 2$. With respect to the canonical decomposition $\mathbb{S} \cong \mathcal{K}^{-1/2} \oplus \mathcal{K}^{1/2}$ these operators have the descriptions

$$\sigma_0 = \left[\begin{array}{cc} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{array} \right]$$

$$\sigma_1 = \left[\begin{array}{cc} 0 & \bar{\varepsilon} \\ -\varepsilon & 0 \end{array} \right]$$

$$\sigma_2 = \left[\begin{array}{cc} 0 & \mathbf{i}\bar{\varepsilon} \\ \mathbf{i}\varepsilon & 0 \end{array} \right]$$

where ε (resp. $\bar{\varepsilon}$) denotes the tensor multiplication by $\bar{\varepsilon}$ (resp ε)

$$\bar{\varepsilon} : \mathcal{K}^{1/2} \to \mathcal{K}^{-1/2} \quad (\text{resp.} \quad \varepsilon : \mathcal{K}^{-1/2} \to \mathcal{K}^{1/2}).$$

If (ω_{ij}) denotes the $\underline{so}(3)$ valued 1-form associated to the Levi-Civita connection via the local frame $\{\zeta_j\}$ i.e.

$$\nabla \zeta_j = \sum_i \omega_{ij} \zeta_i$$

then the canonical connection on $\mathbb{S}$ is defined by

$$\nabla = d - \frac{1}{2} \sum_{i<j} \omega_{ij} \otimes \sigma_i \sigma_j.$$

Using the structural equations (1.3) we deduce that, with respect to the local frame $\{\zeta_j\}$, the canonical connection on $\mathbb{S}$ has the form

$$\hat{\nabla} = \nabla^{\mathbb{S}} = d - \frac{1}{2}(A \otimes \sigma_2 + B \otimes \sigma_1 + C \otimes \sigma_0).$$

Using the fact that N is a Killing m.a.c. we deduce

$$\hat{\nabla}_\zeta = \partial_\zeta - \frac{1}{2} \begin{bmatrix} \mathbf{i}\varphi(x) & 0 \\ 0 & -\mathbf{i}\varphi(x) \end{bmatrix}$$

$$\hat{\nabla}_1 = \partial_{\zeta_1} - \frac{1}{2} \begin{bmatrix} \mathbf{i}C_1 & \lambda\bar{\varepsilon} \\ -\lambda\varepsilon & -\mathbf{i}C_1 \end{bmatrix}$$

$$\hat{\nabla}_2 = \partial_{\zeta_2} - \frac{1}{2} \begin{bmatrix} \mathbf{i}C_2 & \mathbf{i}\lambda\bar{\varepsilon} \\ \mathbf{i}\lambda\varepsilon & -\mathbf{i}C_2 \end{bmatrix}.$$

The canonical, untwisted (geometric) Dirac operator on $\mathbb{S}$ is defined by

$$\mathfrak{D}_0 = \mathfrak{D}_{\mathbb{S}} = \sigma_0 \hat{\nabla}_0 + \sigma_1 \hat{\nabla}_1 + \sigma_2 \hat{\nabla}_2$$

$$= \begin{bmatrix} \mathbf{i}(\partial_\zeta - \mathbf{i}\varphi/2) & \bar{\varepsilon} \otimes \{(\partial_1 + \mathbf{i}C_1/2) + \mathbf{i}(\partial_2 + \mathbf{i}C_2/2)\} \\ \varepsilon \otimes \{-(\partial_1 + \mathbf{i}C_1/2) + \mathbf{i}(\partial_2 + \mathbf{i}C_2/2)\} & -\mathbf{i}(\partial_\zeta + \mathbf{i}\varphi/2) \end{bmatrix} + \lambda \mathbf{1}_{\mathbb{S}}.$$

In terms of the pseudo-Dolbeault operator we have

$$\mathfrak{D}_{\mathbb{S}} = \boldsymbol{\eth}_{\mathbb{S}} + \lambda.$$

More generally if we twist $\mathbb{S}$ by a line bundle L equipped with a connection ∇^L we obtain a geometric Dirac operator on $\mathbb{S}_L = \mathbb{S} \otimes L$ and as above one establishes the following identity

$$\mathfrak{D}_L = \boldsymbol{\eth}_L + \lambda. \tag{2.17}$$

3 The Seiberg-Witten equations

In this section we finally take-up the study of the 3-dimensional Seiberg-Witten equations. We will restrict our considerations to the special case when the 3-manifold N has a(K, λ)-structure. When $\lambda = 0$ it was observed by many authors that these equations can be solved quite explicitly. The situation is more complicated when $\lambda \neq 0$ for the reasons explained in the introduction. We subject N to an anisotropic adiabatic deformation so that in the limit $\lambda_\delta \to 0$ and study the behavior of the solutions of the Seiberg-Witten equations as the metric degenerates. Our study is sufficiently accurate to provide many informations about the exact solutions for the Seiberg-Witten equations corresponding to large δ and a certain range of spin^c structures.

§3.1 Generalities The goal of this subsection is to describe the 3-dimensional Seiberg-Witten equations and then derive a few elementary consequences.

Consider (N, g) a compact, oriented m.a.c 3-manifold. Fix a spin structure on N defined by the square root $\mathcal{K}^{-1/2}$. The data entering the Seiberg-Witten equations are the following.

(a) A spin^c structure determined by the line bundle L.

(b) A connection A of $L \to N$.

(c) A spinor ϕ i.e. a section of the complex spinor bundle $\mathbb{S}_L$ associated to the given spin^c structure.

The connection A defines a geometric Dirac operator $\mathcal{D}_A$ on $\mathbb{S}_L$. The Seiberg-Witten equations are

$$\left\{ \begin{array}{rcl} \mathcal{D}_{\mathbf{A}}\phi & = & 0 \\ \mathbf{c}(*F_{\mathbf{A}}) & = & \tau(\phi) \end{array} \right.$$

where $*$ is the Hodge $*$-operator of the metric g, τ is defined in (2.6) and $\mathbf{c}$ is defined in (2.5). We will omit the symbol $\mathbf{c}$ when no confusion is possible.

The Seiberg-Witten equations have a variational nature. Fix a smooth connection A_0 on L and define

$$\mathfrak{f} : L^{1,2}(\mathbb{S}_L \oplus \mathbf{i}T^*N) \to \mathbb{R}$$

by

$$\mathfrak{f}(\psi, a) = \frac{1}{2}\int_N a \wedge (F_{A_0} + F_{A_0+a}) + \frac{1}{2}\int_N \langle \psi, \mathcal{D}_{A_0+a}\psi \rangle dv_g.$$

Lemma 3.1 The differential of $\mathfrak{f}$ at a point $\mathfrak{c} = (\phi, a)$ is

$$d_{\mathfrak{c}}\mathfrak{f}(\dot{\phi}, \dot{a}) = \int \langle \dot{a}, \mathbf{c}^{-1}(\tau(\phi)) - *F_{A_0+a} \rangle dv_g + \int_N \mathfrak{Re}\langle \dot{\phi}, \mathcal{D}_{A_0+a}\phi \rangle dv_g.$$

Proof Set $A = A_0 + a$. We have

$$d_{gc}\mathfrak{f}(\dot\phi, \dot a) = \frac{d}{dt}|_{t=0}\ \mathfrak{f}(\phi + t\dot\phi, a + t\dot a)$$

$$= \frac{1}{2}\int_N \dot a \wedge (F_{A_0} + F_A) + \frac{1}{2}\int_N a \wedge d\dot a - \int_N \mathfrak{Re}\langle \dot\phi, \mathfrak{D}_A\phi\rangle dv_g - \frac{1}{2}\int_N \langle \phi, \mathbf{c}(\dot a)\phi\rangle dv_g$$

$$\text{(Stokes)} = \frac{1}{2}\int_N \dot a \wedge (F_{A_0} + F_A) + \frac{1}{2}\int_N \dot a \wedge da - \frac{1}{2}\int_N \langle \phi, \mathbf{c}(\dot a)\phi\rangle dv_g - \int_N \mathfrak{Re}\langle \dot\phi, \mathfrak{D}_A\phi\rangle dv_g$$

$$\int_N \dot a \wedge F_A + \frac{1}{2}\int_N \langle \phi, \mathbf{c}(\dot a)\phi\rangle dv_g + \int_N \mathfrak{Re}\langle \dot\phi, \mathfrak{D}_A\phi\rangle dv_g.$$

Since both $\dot a$ and F_A are purely imaginary we have is

$$\int_N \langle \dot a, *F_A\rangle dv_g = -\int_N \dot a \wedge F_A$$

where $*$ is the *complex linear* Hodge $*$-operator. On the other hand, a simple computation shows that ($\dot a = \sum_i \dot a_i \eta_i$)

$$\int_N \langle \phi, \mathbf{c}(\dot a)\phi\rangle dv_g = -\int_N \sum_i \dot a_i \langle \phi, \mathbf{c}(\eta_i)\phi\rangle dv_g$$

$$= -\int_N \sum_i \langle\, \dot a_i \eta_i, \overline{\langle \phi, \mathbf{c}(\eta_i)\phi\rangle}\, \eta_i\,\rangle dv_g = 2\int_N \langle \dot a, \mathbf{c}^{-1}(\tau(\phi))\rangle dv_g.$$

Putting all the above together we get the lemma. $\square$

The gauge group $\mathfrak{G}_L = \mathrm{Aut}\,(L) \cong \mathrm{Map}\,(N, S^1)$ acts on the space of pairs (ψ, a) by

$$\gamma \cdot (\psi, a) := (\gamma \cdot \psi, a - \gamma^{-1}d\gamma).$$

Moreover

$$\mathfrak{f}(\gamma \cdot (\psi, a)) - \mathfrak{f}(\psi, a) = -\int_N \gamma^{-1}d\gamma \wedge F_{A_0} = 2\pi\mathbf{i}\int_N \gamma^{-1}d\gamma \wedge c_1(A_0)$$

Thus $\mathfrak{f}$ is unchanged by the gauge transformations homotopic to the constants. We see that the critical points of $\mathfrak{f}$ are precisely the solutions of the Seiberg-Witten equations. In particular, the above considerations show that the moduli space of solutions is invariant under the action of $\mathfrak{G}_L$ and so it suffices to look at the quotient of this action.

The Seiberg-Witten equations have a more explicit description once we choose an adapted orthonormal frame ζ, ζ_1, ζ_2. Using the decomposition $\mathbb{S}_L \cong \mathcal{K}^{-1/2} \otimes L \oplus \mathcal{K}^{1/2} \otimes L$ we can represent ϕ as

$$\phi = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

If N is a (K,λ)- manifold and we denote by $\nabla^{\pm}$ the covariant derivatives induced by A on $\mathcal{K}^{\pm 1/2}\otimes L$ then the Seiberg-Witten equations can be rephrased as

$$\begin{cases} \mathbf{i}\nabla^-_\zeta\alpha & +{}^\flat\overline{\nabla}^+\beta & +\lambda\alpha & = & 0 \\ ({}^\flat\overline{\nabla}^+)^*\alpha & -\mathbf{i}\nabla^+_\zeta\beta & +\lambda\beta & = & 0 \\ & & \frac{1}{2}(|\alpha|^2-|\beta|^2) & = & \mathbf{i}F_{\nabla}(\zeta_1,\zeta_2) \\ & & \mathbf{i}\alpha\bar{\beta} & = & \bar{\varepsilon}\otimes F_{\nabla}(\zeta_1+\mathbf{i}\zeta_2,\zeta) = -F_A^{0,1} \end{cases} \tag{3.1}$$

In particular, note that

$$\eta\wedge c_1(A) = \frac{\mathbf{i}}{2\pi}F_A(\zeta_1,\zeta_2)\eta\wedge\eta^1\wedge\eta^2 = \frac{1}{4\pi}(|\alpha|^2-|\beta|^2)dvol_g.$$

Hence

$$\int_N \eta\wedge c_1(F_{\mathbf{A}}) = \frac{1}{4\pi}(\|\alpha\|^2-\|\beta\|^2) \tag{3.2}$$

where $\|\cdot\|$ denotes the L^2- norm over N.

In terms of the operators Z_A and T_A defined in §2.3 we can rewrite the first two equations as

$$(Z_A+T_A)\phi = -\lambda\phi.$$

The anti-commutation relation (2.16) shows that if (A,ϕ) is a solution of the Seiberg-Witten equation then

$$(Z_A+T_A)^2\phi = Z_A^2\phi + T_A^2\phi - \lambda T_A\phi + \left[\begin{array}{cc} 0 & \alpha\bar{\beta} \\ \bar{\alpha}\beta & 0 \end{array}\right]\cdot\left[\begin{array}{c} \alpha \\ \beta \end{array}\right]. \tag{3.3}$$

§3.2 Adiabatic limits We now have all the data we need to study the behavior of the Seiberg-Witten equations as the metric is anisotropically deformed until it degenerates.

Let $(N.\eta,g)$ and $\mathcal{K}^{1/2}$ as above. As usual we denote by (N,η_δ,g_δ) the anisotropic deformation defined in §1.1. For each $\delta\geq 1$ we we will refer to the Seiberg-Witten equations defined in terms of the metric g_δ as SW_δ equations. More explicitly these are $(|\zeta|_{g_1}\equiv 1,\ \ |\eta|_{g_1}\equiv 1)$

$$\begin{cases} \delta\mathbf{i}\nabla^-_\zeta\alpha & +{}^\flat\overline{\nabla}^+\beta & +\frac{\lambda}{\delta}\alpha & = & 0 \\ ({}^\flat\overline{\nabla}^+)^*\alpha & -\delta\mathbf{i}\nabla^+_\zeta\beta & +\frac{\lambda}{\delta}\beta & = & 0 \\ & & \frac{1}{2}(|\alpha|^2-|\beta|^2) & = & \mathbf{i}F_{\nabla}(\zeta_1,\zeta_2) \\ & & \mathbf{i}\alpha\bar{\beta} & = & \delta\bar{\varepsilon}\otimes F_{\nabla}(\zeta_1+\mathbf{i}\zeta_2,\zeta) \end{cases} \tag{3.4}$$

The operator Z_A depends on the metric g_δ through the Levi-Civita connection $\nabla^\perp$ on $\mathcal{K}$ and so we will write $Z_{A,\delta}$ to emphasize this dependence. As $\delta \to \infty$ we have

$$\frac{1}{\delta} Z_{A,\delta} \to Z_{A,\infty} := \left[\begin{array}{cc} \mathbf{i}\hat{\nabla}_\zeta^{A,-} & 0 \\ 0 & -\mathbf{i}\hat{\nabla}_\zeta^{A,+} \end{array} \right]$$

where for each connection A on L we denoted by $\hat{\nabla}^{A,\pm}$ the connection on $\mathcal{K}^{\pm 1/2} \otimes L$ obtained by tensoring the limiting connection ∇^∞ on $\mathcal{K}^{\pm 1/2}$ (described in (1.13)) with the connection A. On the other hand, T_A is unaffected by the adiabatic changes in the metric.

We will fix a smooth connection ∇^0 on L. The Sobolev norms will be defined in terms of this connection and its tensor products with the connections induced by the Levi-Civita connection of the *fixed* metric $g = g_1$. An arbitrary connection on L will have the form

$$\nabla^0 + A_\delta, \quad A_\delta \in \underline{u}(1) \otimes \Omega^1(N).$$

For $1 \leq p \leq \infty$ we denoted by $\|\cdot\|_p$ the L^p-norm with respect to the metric $g = g_1$, by $\|\cdot\|$ the L^2-norm with respect to the same metric while $\|\cdot\|_\delta$ will denote the L^2-norm with respect to the metric g_δ).

We denote by $\mathfrak{A}_L$ the affine space of *smooth* connections on L and by $\mathcal{S}_\delta$ the collection of gauge equivalence classes of solutions of SW_δ.

Theorem 3.2 Let (N, η, g) as above and fix a spin^c structure defined by a complex line bundle L (so that the associated complex spinor bundle has determinant L^2). Assume that for each sufficiently large δ

$$\mathcal{S}_\delta \neq \emptyset.$$

Then any sequence $\{([A_\delta], [\phi_\delta]) \in \mathcal{S}_\delta\ ;\ \delta \gg 1\}$ admits a subsequence which converges in the $L^{1,2}$ topology to a pair

$$([A], [\phi]) \in (\mathfrak{A}_L \times C^\infty(L))/\mathrm{Aut}\,(L)$$

satisfying the following conditions.

$$F_A(\zeta, \cdot) \equiv 0 \tag{3.5}$$

$$^\flat\overline{\nabla}^+ \beta = \hat{\nabla}_\zeta^{A,-} \alpha = 0 \tag{3.6}$$

$$(^\flat\overline{\nabla}^+)^* \alpha = \hat{\nabla}_\zeta^{A,+} \beta = 0 \tag{3.7}$$

$$\|\alpha\| \cdot \|\beta\| = 0. \tag{3.8}$$

The equations (3.6) and (3.7) can be equivalently rephrased as

$$Z_{A,\infty}\phi = T_A\phi = 0. \tag{3.9}$$

We can say that the Seiberg-Witten equations decouple in the adiabatic limit.

Proof The proof of the theorem relies essentially on the following *uniform* estimates.

Lemma 3.3 There exist $R_1,\ R_2 > 0$ such that

$$\sup|\phi_\delta(x)| \leq R_1 \ \ \forall \delta \geq 1 \tag{3.10}$$

and

$$\|F(A_\delta)\|_\infty \leq R_2 \ \ \ \forall\delta \geq 1. \tag{3.11}$$

$$\|\,|\alpha_\delta| \,\cdot\, |\beta_\delta|\,\|_2 = O(1/\delta) \quad \text{as} \ \ \delta \to \infty. \tag{3.12}$$

Proof of the lemma As in Lemma 2 of [KM] we deduce that

$$\sup|\phi_\delta(x)| \leq \sup|s_\delta(x)|$$

where s_δ denotes the scalar curvature of g_δ. The estimate (3.10) is now a consequence of (1.16) in §1.2.

To prove (3.11) note first that $F(A_\delta)$ splits into two orthogonal parts. A horizontal part

$$F^h(A_\delta) = F_{12}(A_\delta)\eta^1 \wedge \eta^2$$

and a vertical part

$$F^v(A_\delta) = F_{01}(A_\delta)\eta \wedge \eta^1 + F_{20}(A_\delta)\eta^2 \wedge \eta.$$

The third equation in (3.4) coupled with (3.10) yields

$$\|F^h(A_\delta)\|_\infty = O(1) \ \ \text{as} \ \ \delta \to \infty.$$

The fourth equality in (3.4) implies

$$\|F^v(A_\delta)\|_\infty \leq \text{const.}\delta^{-1}\|\,|\alpha_\delta| \cdot |\beta_\delta|\,\|_\infty = O(\delta^{-1}) \tag{3.13}$$

where

$$\phi_\delta = \begin{bmatrix} \alpha_\delta \\ \beta_\delta \end{bmatrix}.$$

To prove (3.12) we will use (3.3) and we get (writing Z_{A_δ} instead of the more accurate $Z_{A_\delta,\delta}$)

$$\lambda_\delta^2\phi_\delta = (Z_{A_\delta} + T_{A_\delta})^2\phi_\delta = Z_{A_\delta}^2\phi_\delta + T_{A_\delta}^2\phi_\delta - \lambda_\delta T_{A_\delta}\phi_\delta + \left[\begin{array}{cc} 0 & \alpha_\delta\bar{\beta}_\delta \\ \bar{\alpha_\delta}\beta_\delta & 0 \end{array}\right]\left[\begin{array}{c} \alpha_\delta \\ \beta_\delta \end{array}\right].$$

Taking the inner product with ϕ_δ in the above equality and then integrating by parts with respect to the metric g_δ we deduce

$$\|Z_{A_\delta}\phi_\delta\|_\delta^2 + \|T_{A_\delta}\phi_\delta\|_\delta^2 + 2\|\,|\alpha_\delta|\,\cdot\,|\beta_\delta|\,\|_\delta^2 = \lambda_\delta^2\|\phi_\delta\|_\delta^2 + \lambda_\delta\langle T_{A_\delta}\phi_\delta, \phi_\delta\rangle_\delta.$$

The Cauchy inequality yields

$$\|Z_{A_\delta}\phi_\delta\|_\delta^2 + \|T_{A_\delta}\phi_\delta\|_\delta^2 + 2\|\,|\alpha_\delta|\,\cdot\,|\beta_\delta|\,\|_\delta^2 \leq |\lambda_\delta|\cdot\|\phi_\delta\|_\delta\,(|\lambda_\delta|\cdot\|\phi_\delta\|_\delta + \|T_{A_\delta}\phi_\delta\|_\delta) \tag{3.14}$$

Since $|\lambda_\delta|,\ \|\phi_\delta\|_\delta^2 = O(\delta^{-1})$ we first deduce from the above inequality that $\|T_{A_\delta}\phi_\delta\|_\delta = O(\delta^{-3/2})$ and then

$$\|\,|\alpha_\delta|\,\cdot\,|\beta_\delta|\,\|_\delta = O(\delta^{-3/2}).$$

Up to a rescaling this is precisely the inequality (3.12). □

From the estimate (3.11) we deduce that A_δ (modulo Coulomb gauges) is bounded in the $L^{1,p}$-norm for any $1 < p < \infty$. Thus a subsequence converges strongly in L^p and weakly in $L^{1,p}$ to some Hölder continuous connection A on L. The fourth equation in (3.4) and the estimate (3.12) we have just proved show that

$$\|F_{A_\delta}^{0,1}\| = O(\delta^{-2}).$$

The condition (3.9) is a consequence of the following auxiliary result.

Lemma 3.4 (Adiabatic decoupling lemma) Consider a sequence of smooth connections $A_\delta \in \mathfrak{A}_L$ satisfying the following conditions.
(i) A_δ converges in the weak $L^{1,p}$ topology ($p > 3 = \dim N$) to a Hölder continuous connection A.
(ii) $\|F_{A_\delta}^{0,1}\| = o(\delta^{-1})$ as $\delta \to \infty$.

Then any sequence $\phi_\delta \in C^\infty(\mathbb{S}_L)$ such that

$$\|\phi_\delta\| = O(1) \ \text{ as } \delta \to \infty$$

and

$$\mathcal{D}_{A_\delta}\phi_\delta = 0$$

contains a subsequence which converges strongly in $L^{1,2}(\mathbb{S}_L)$ to a spinor $\phi \in L^{1,2}(\mathbb{S}_L)$ satisfying

$$Z_{A,\infty}\phi = T_A\phi = 0. \tag{3.15}$$

Proof Set $Z_{A_\delta} = Z_{A_\delta,\delta}$ and

$$\Xi_\delta = \frac{1}{\delta} Z_{A_\delta}$$

The coefficients of Ξ_δ converge uniformly to the coefficients of $Z_{A,\infty}$. More precisely, the condition (i) implies $\Xi_\delta = Z_{A,\infty} + R$ where the zeroth order term R satisfies

$$\|R\|_\infty = o(1) \tag{3.16}$$

Note that although Ξ_δ is defined using the metric g_δ it is formally self-adjoint with respect to the metric g_1 as well.

The equation $\mathcal{D}_{A_\delta}\phi_\delta = 0$ can be rewritten as

$$(Z_{A_\delta} + T_{A_\delta})\phi_\delta = -\lambda_\delta\phi_\delta.$$

Using the equation (2.16) (with the background m.a.c. structure $(g_\delta, \zeta_\delta = \delta\zeta)$) we get

$$\lambda_\delta^2\phi_\delta = (Z_{A_\delta}^2 + T_{A_\delta})^2\phi_\delta = -\lambda_\delta T_{A_\delta} + \mathcal{F}_\delta \tag{3.17}$$

where

$$\mathcal{F}_\delta = -\mathbf{i}\delta \left[\begin{array}{cc} 0 & -\bar{\varepsilon} F_{A_\delta}(\zeta, \zeta_1 + \mathbf{i}\zeta_2) \\ \varepsilon F_{A_\delta}(\zeta, \zeta_1 - \mathbf{i}\zeta_2) & 0 \end{array} \right] = -\mathbf{i}\delta \left[\begin{array}{cc} 0 & -F_{A_\delta}^{0,1} \\ F_{A_\delta}^{1,0} & 0 \end{array} \right].$$

Take the inner product with ϕ_δ of both sides in (3.17) and then integrate by parts using *the fixed metric* g_1. We get

$$\delta^2\|\Xi_\delta\phi_\delta^2\|^2 + \|T_{A_\delta}\phi_\delta\|^2 \leq \|\phi_\delta\| \left\{ |\lambda_\delta| \cdot \|T_{A_\delta}\| + \|\mathcal{F}_\delta\| + \lambda_\delta^2\|\phi_\delta\| \right\}.$$

Since $\|\phi_\delta\| = O(1)$ and $\|\mathcal{F}_\delta\| = O(\delta\|F_{A_\delta}^{0,1}\|) = o(1)$ we deduce

$$\delta^2\|\Xi_\delta\phi_\delta^2\|^2 + \|T_{A_\delta}\phi_\delta\|^2 \leq C\left(\delta^{-1}\|T_{A_\delta}\phi_\delta\| + \delta^{-2} + o(1)\right).$$

This yields

$$\|T_{A_\delta}\phi_\delta\| = o(1) \ \text{ and } \ \|\Xi_\delta\phi_\delta\| = o(\delta^{-2}). \tag{3.18}$$

Set $\mathcal{D}_\infty = Z_{A,\infty} + T_A$. We know have

$$\mathcal{D}_\infty\phi_\delta = \psi_\delta := \Xi_\delta\phi_\delta + T_{A_\delta}\phi_\delta - R\phi_\delta$$

The estimates (3.16) and (3.18) show that $\|\psi_\delta\| = o(1)$. The elliptic estimates applied to the *fixed* elliptic operator $\mathcal{D}_\infty$ show that

$$\|\phi_\delta\|_{1,2} \leq C(\|\phi_\delta\| + \|\psi\|_\delta).$$

Thus ϕ_δ is bounded in the $L^{1,2}$-norm so a subsequence (still denoted by ϕ_δ) will converge strongly in L^2. Using again the elliptic estimates we get

$$\|\phi_{\delta_1} - \phi_{\delta_2}\|_{1,2} \leq C(\|\phi_{\delta_1} - \phi_{\delta_2}\| + \|\psi_{\delta_1}\| + \|\psi_{\delta_2}\|).$$

This shows the (sub)sequence ϕ_δ is Cauchy in the $L^{1,2}$ norm and in particular it must converge in this norm to a ϕ which must satisfy

$$Z_{A,\infty}\phi = T_A\phi = 0$$

due to (3.18). The proof is complete. $\square$

Remark 3.5 A result stronger than stated above is true. If the connections A_δ are as in the adiabatic decoupling lemma and $\psi_\delta \in L^{1,2}(\mathbb{S})$ are such that $\|\psi_\delta\| = 1$ and $\|\mathcal{D}_\delta\psi_\delta\| = o(1)$ then a subsequence of ψ_δ converges strongly in $L^{1,2}$ to a spinor ψ satisfying (3.15).

We can now conclude the proof of the theorem. From the above lemma we know that ϕ_δ converges in $L^{1,2}$. By Sobolev inequality the sequence ϕ_δ must also converge in L^p, $1 \leq p \leq 6$. The last two equations in (3.4) allow us to conclude that F_{A_δ} is actually convergent in L^3 so we can conclude that modulo Coulomb gauges the connections A_δ converge *strongly* in $L^{1,3}$.

Note that ϕ satisfies a condition slightly weaker than (3.8) namely

$$\|\, |\alpha| \,\cdot\, |\beta|\, \| = 0.$$

We will now prove this implies (3.8). We can rewrite the limiting connection $\hat{\nabla}^A$ as a sum $\hat{\nabla}^A = \nabla^0 + B$, where $B \in L^{1,p}(\mathrm{End}\,(\mathbb{S}_L))$ for any $1 \leq p < \infty$. Note that α and β satisfy the elliptic Dirac equation with Hölder continuous coefficients

$$\mathcal{D}_\infty\phi = 0 \tag{3.19}$$

This equation can be rewritten as

$$\mathcal{D}^0_\nabla\phi + Q_B\phi = 0 \tag{3.20}$$

where $Q_B \in L^{1,p}(\mathrm{End}\,(\mathbb{S}_L))$, $1 \leq p < \infty$. Since $\phi \in L^{1,2}(\mathbb{S}_L) \cap L^\infty(\mathbb{S}_L)$ we deduce

$$Q_B\phi \in L^{1,2}(\mathbb{S}_L).$$

Using this in (3.20) we deduce $\phi \in L^{2,2}$. Note that $\hat{\nabla}^\pm_\zeta = \nabla^{A,\pm}_\zeta \pm \lambda/2$ where ∇^A denotes the spinor connection on $\mathbb{S}_L$ obtained by tensoring the connection A with the Levi-Civita connection defined by the metric g_1. Applying ${}^\flat\overline{\nabla}^+$ to the second equation in (3.19) and using the commutator equality (2.14) we deduce after some simple manipulations that

$${}^\flat\overline{\nabla}{}^\flat\overline{\nabla}^*\alpha - \nabla^2_\zeta\alpha + |\beta|^2\alpha + \lambda{}^\flat\overline{\nabla}\beta = 0.$$

Since ${}^\flat\overline{\nabla}\beta = 0$ and $\alpha \otimes \beta \equiv 0$ we deduce

$${}^\flat\overline{\nabla}{}^\flat\overline{\nabla}^*\alpha - \nabla^2_\zeta\alpha = 0.$$

We can rewrite the above equation as

$$ {}^\flat\bar{\nabla}^0({}^\flat\bar{\nabla}^0)^*\alpha - \nabla^2_\zeta\alpha + P(\nabla\alpha) + Q\alpha = 0 $$

where the "coefficient" $P \in L^{1,2} \subset L^6$ while $Q \in L^2$ is at least L^2 (it is a linear combination of the connection A and its derivatives) while the differential operator above is a generalized Laplacian. This is more than sufficient to apply the unique continuation principle in Thm. 4.3 of [H] to deduce that if α vanishes on an open subset of N then it must vanish everywhere.

A dual argument proves a similar result for β. Since the product $\alpha\otimes\beta$ is identically zero we deduce that one of them must vanish on an open set and hence everywhere. The equality (3.8) is proved and this completes the proof of the theorem. $\square$

§3.3 The Seiberg-Witten equations on circle bundles When N is the total space of a circle bundle over a surface the above result can be given a more precise form.

Let N_ℓ be the total space of a degree ℓ principal S^1 bundle

$$ S^1 \hookrightarrow N_\ell \xrightarrow{\pi} \Sigma $$

where Σ is a compact surface of genus $g \geq 1$. Fix a complex structure on Σ and denote by K the canonical line bundle. Then

$$ \mathcal{K} \cong \pi^* K. $$

Moreover, according to (1.22), in this case the limiting connection ∇^∞ on $\mathcal{K}$ coincides with the connection pulled back from K.

Now fix a spin structure on Σ by choosing a square root $K_\Sigma^{1/2}$ on Σ. This defines by pullback a spin structure on N. Denote by $(N_\ell, \eta_\delta, h_\delta)$ the Boothby-Wang (K, λ) structure described in §1.3.

Fix a spin^c structure on N given by a line bundle L_N. If the Seiberg-Witten equations SW_δ corresponding to the above spin^c structure have solutions for every $\delta \gg 1$ we deduce from the above theorem that L_N admits a connection A such that $F_A(\zeta, \cdot) \equiv 0$. This implies $\pi_* F_A \equiv 0$ where π_* denotes the integration along the fibers of $\pi : N_\ell \to \Sigma$. From the Gysin sequence of this fibration we deduce that $c_1(L_N) \in \pi^* H^2(\Sigma)$ i.e. L_N is the pullback of a line bundle $L_\Sigma \to \Sigma$. In particular we have the following vanishing result.

Corollary 3.6 Fix a spin^c structure on N_ℓ which is not the pullback of any spin^c structure on Σ. Then the Seiberg-Witten equations SW_δ have no solutions for any $\delta \gg 1$. In particular, when $|\ell| = 1$ then for any *nontrivial* spin^c structure on N and for all $\delta \gg 1$ the equations SW_δ have no solutions.

Now fix a spin^c structure on N_ℓ defined by a complex line bundle $L_N = \pi^* L_\Sigma$. Any of adiabatic limit of the solutions of Seiberg-Witten equation is a pair (A, ϕ) satisfying the following conditions.
(a) A connection A on L_N with horizontal curvature i.e. $F_A(\zeta, \cdot) \equiv 0$.
(b) Sections $\alpha \in \mathcal{K}^{-1/2} \otimes L_N$ and $\beta \in \mathcal{K}^{1/2} \otimes L_N$ such that

$$Z_{A,\infty}\phi = T_A\phi = 0 \Longleftrightarrow {}^\flat\bar{\nabla}^A\beta = 0 = \hat{\nabla}^A_\zeta\alpha \;\text{ and }\; ({}^\flat\bar{\nabla}^A)^*\alpha = 0 = \hat{\nabla}^A_\zeta\beta.$$

$$\|\alpha\| \cdot \|\beta\| = 0$$

where the derivation $\hat{\nabla}^A_\zeta$ is the tensor product of the corresponding derivations on $\pi^* K_\Sigma^{\pm 1/2}$ and L_N.
(c) $\frac{1}{2}(|\alpha|^2 - |\beta|^2) = \mathbf{i} F_A(\zeta_1, \zeta_2)$.

We will refer to the conditions (a)-(c) above as the *adiabatic Seiberg-Witten equations*. A pair (A, ϕ) satisfying these conditions will be called an *adiabatic solution* of the (adiabatic) Seiberg-Witten equations. We denote by $\mathcal{A}_\infty = \mathcal{A}_\infty(\ell, g, L_N)$ the collection of gauge equivalence classes of adiabatic solutions. Note that it is not a priori clear whether any adiabatic solution is in fact an adiabatic limit. All we can state at this point is that

$$\mathcal{S}_\infty := \lim \mathcal{S}_\delta \subset \mathcal{A}_\infty.$$

To make further progress understanding the nature of the adiabatic solutions we need to distinguish two situations.

A. Reducible adiabatic solutions i.e. $\|\alpha\| + \|\beta\| = 0$. In this case the connection A must be flat. Thus it is uniquely defined by its holonomy representation

$$\mathrm{hol}_A : \pi_1(N_\ell) \to S^1.$$

The fundamental group of N_ℓ can be presented as

$$\pi_1(N_\ell) = \langle a_1, b_1, \dots a_g, b_g, f | f^{-\ell} \cdot \prod_{i=1}^{g} [a_i, b_i] = [a_j, f] = [b_k, f] = 1, \;\; \forall j, k \rangle$$

so that the space of its S^1 representations is a $2g+1$-dimensional torus if $\ell = 0$ while if $\ell \neq 0$ it is a collection of $|\ell|$ tori T^{2g} parameterized by the group of $|\ell|$-th roots of unity. For any representation $\rho : \pi_1(N) \to S^1$ we have $\rho(f) = \exp(2\pi k \mathbf{i}/\ell)$ for some $k \in \mathbb{Z}$ and if we denote by $L_{N,\rho}$ the line bundle it determines on N we have

$$c_1(L_{N,\rho}) = \hat{k} \in \mathbb{Z}_{|\ell|} \subset H^2(N_\ell, \mathbb{Z}). \tag{3.21}$$

To see this, denote by A_ρ flat the connection on $L_{N,\rho}$ with holonomy ρ and set $B_\rho := A_\rho + \mathbf{i}k/\ell\eta$. Then the curvature of B_ρ is purely horizontal, $F_{B_\rho} = \mathbf{i}k/\ell d\eta$ and its holonomy along the fibers is zero. Thus B_ρ is the pull-back of a connection D_ρ on a line bundle $L_{\Sigma,\rho} \to \Sigma$ whose curvature satisfies

$$\pi^* F_{D_\rho} = \mathbf{i}k/\ell d\eta = -2\mathbf{i}k *_N \eta = -2\mathbf{i}k\pi^* d\,\mathrm{vol}_\Sigma.$$

Thus

$$c_1(D_\rho) = k/\pi \, d\,\mathrm{vol}_\Sigma$$

i.e. $\deg L_{\Sigma,\rho} = k$ since by construction $\mathrm{vol}\,(\Sigma) = \pi$. The equality (3.21) now follows from the equality $L_{N,\rho} = \pi^* L_{\Sigma,\rho}$ combined with Gysin's exact sequence. To summarize, when $c_1(L_N)$ is a torsion class, the collection of gauge equivalence classes of reducible adiabatic solutions forms is a torus of dimension $2g$.

B. Irreducible adiabatic solutions i.e. $\|\alpha\| + \|\beta\| \neq 0$.

Lemma 3.7 If $\|\alpha\| + \|\beta\| \neq 0$ then (modulo gauge transformations) the connection A is the pull back of a connection on a line bundle $L_\Sigma \to \Sigma$.

Proof of the lemma Because the connection A is vertically flat one can prove easily that the holonomy along a fiber is independent of the particular fiber. Since either α or β is not identically zero one deduces from the condition (b) above that this holonomy is trivial. □

Let (A, α, β) adiabatic limits as in the above lemma. Thus there exists a connection $\underline{A}$ on L_Σ such that $\pi^* L_\Sigma = L_N$ and $\pi^* \underline{A} = A$. Since α and β are covariant constant along fibers they can be regarded as sections of $L_\Sigma \otimes K_\Sigma^{\pm 1/2}$. We can decide which of α or β vanishes. More precisely if

$$\deg L_\Sigma < \deg K_\Sigma^{-1/2} = 1 - g$$

then the line bundle $L_\Sigma \otimes K_\Sigma^{1/2}$ cannot admit holomorphic sections so that $\beta \equiv 0$. In particular, $\alpha \not\equiv 0$ In this case, using the equality (3.2) we deduce after an integration along fibers that

$$1 - g > \deg L_\Sigma = \frac{\mathbf{i}}{2\pi} \int_N \eta \wedge F_A = \frac{1}{2} \|\alpha\|^2.$$

This is impossible since $\alpha \neq 0$. If

$$\deg L_\Sigma > \deg K_\Sigma^{1/2} = g - 1$$

then $L_\Sigma \otimes K_\Sigma^{-1/2}$ cannot admit antiholomorphic sections so that $\alpha \equiv 0$. Reasoning as above we deduce another contradiction so we can conclude the adiabatic limit set is empty when $|\deg L_\Sigma| > g - 1$.

We now analyze what happens when $|\deg L_\Sigma| \leq g - 1$. We will discuss only the case $1 - g \leq \deg L_\Sigma \leq 0$. The other half is completely similar.

Using the equality (3.2) again we deduce

$$\frac{1}{4\pi}(\|\alpha\|^2 - \|\beta\|^2) = \deg L_\Sigma \leq 0.$$

Since one of α or β is identically zero we deduce that $\alpha \equiv 0$. Thus the adiabatic limit set consists of *normalized holomorphic pairs* i.e pairs of the form

$$(\text{holomorphic structure on } L_\Sigma\,,\ \text{holomorphic section } \beta \text{ of } L_\Sigma \otimes K_\Sigma^{1/2})$$

where β is normalized by

$$\|\beta\|_\Sigma^2 = 2|\deg L_\Sigma| \tag{3.22}$$

Note that when $\deg L_\Sigma = 0$ we are actually in case **A** so it should be excluded from this discussion. When $\deg L_\Sigma = 1-g$ the bundle $L_\Sigma \otimes K_\Sigma^{1/2}$ is topologically trivial. The only topologically trivial holomorphic line bundle which admits a holomorphic section is the *holomorphically trivial* line bundle. Hence, when $\deg L_\Sigma = 1-g$ there exists exactly one irreducible adiabatic solution (up to a gauge transformations).

The above picture resembles very much the exact computations in the trivial case $\ell = 0$. There are however some notable differences. When $\ell = 0$ if the spin^c structure is non-trivial there exist no reducible solutions for the simple fact that there exist no flat connections. This is no longer the case when $\ell \neq 0$. Moreover, according to Gysin's exact sequence the kernel of the morphism

$$\pi^* : \mathbb{Z} \cong H^2(\Sigma, \mathbb{Z}) \to H^2(N_\ell, \mathbb{Z})$$

is the subgroup $\ell\mathbb{Z}$. If $\ell = 0$ this means the spin^c-structure L_N uniquely determines a line bundle $L_\Sigma \to \Sigma$ such that $\pi^* L_\Sigma \cong L_N$.

If $\ell \neq 0$ there are infinitely many line bundles on Σ with the above property and their degrees are congruent modulo ℓ with $c_1(L_N) \in \mathbb{Z}_\ell \subset H^2(N_\ell, \mathbb{Z})$. On the other hand, in case **B** the only line bundles on Σ relevant in the adiabatic limit are those of degree in the interval $[-(g-1), (g-1)]$.

Assume now $|\ell| \geq 2g$ and

$$c_1(L_N) \in \mathcal{V}_{g,\ell} = \{g, g+1, \cdots, |\ell| - g\} \pmod{\ell} \subset \mathbb{Z}_\ell$$

Then there is no integer n such that $n \bmod \ell \in \mathcal{V}_{g,\ell}$ and $0 < |n| \leq g-1$. In other words if $c_1(L_N) \in \mathcal{V}_{g,\ell}$ then there exists no line bundle $L_\Sigma \to \Sigma$ of degree $0 < |\deg L_\Sigma| \leq g-1$ such that $\pi^* L_\Sigma = L_N$. We have thus proved the following result.

Corollary 3.8 (a) If $|\ell| \geq 2g \geq 4$ and $L_N \to N_\ell$ is a line bundle on N_ℓ with $c_1(L_N) \in \mathcal{V}_{g,\ell}$ then the collection $\mathcal{A}_\infty$ of adiabatic solutions corresponding to the spin^c structure defined by L_N consists only of reducible configurations i.e. pairs

$$(\text{flat connections},\ \text{zero spinor}).$$

(b) If $g = 1$ i.e. Σ is a torus, then the set $\mathcal{A}_\infty$ of adiabatic solutions corresponding to *any* pulled-back spin^c structure consists only of reducible configurations.

Any reducible adiabatic solution is obviously a bona-fide solution of SW_δ for all $\delta > 0$. Thus for the spin^c structures in the above corollary we have trivially

$$\mathcal{S}_\infty = \mathcal{A}_\infty$$

In these cases we have $\mathcal{S}_\infty \subset \mathcal{S}_\delta$ and it is thus natural to ask whether this is a *strict* inclusion for *all* $\delta \gg 1$ i.e. whether there exist adiabatic solutions which are limits of *irreducible* solutions. The estimating techniques of [GM] (see also [Ber]) show that for sufficiently large δ if (A, ϕ) is a solution of the Seiberg-Witten equation SW_δ on a circle bundle over a *flat torus* then $\dim \ker \mathfrak{D}_A \leq 2$. This seems to indicate that the chances that in this case nontrivial solutions exist are very slim. More generally we can ask whether $\mathcal{S}_\infty$ does indeed coincide with $\mathcal{A}_\infty$ and if this is the case how can one use the explicit knowledge of $\mathcal{A}_\infty$ to obtain better informations about $\mathcal{S}_\delta$ with large δ.

A first step in this direction is contained in the following result.

Proposition 3.9 Fix a spin^c structure on $N_{g,\ell}$ defined by a line bundle $L_N \to N$ such that $c_1(L_N) \in \mathcal{V}_{g,\ell}$. Then

$$\mathcal{A}_\infty(g, \ell, L_N) = \mathcal{S}_\delta(g, \ell, L_N), \quad \forall \delta \gg 1.$$

The range $\mathcal{V}_{g,\ell}$ of spin^c structures will be called the *adiabatically stable range*.

Proof Assume the contrary. Thus, for all $\delta \gg 1$ there exists $(A_\delta, \phi_\delta) \in \mathcal{S}_\delta$ such that $\phi_\delta \neq 0$. Set $\psi_\delta = \frac{1}{\|\phi_\delta\|}\phi_\delta$ so that $\|\psi_\delta\| = 1$, $\forall \delta$. We can now apply the adiabatic decoupling lemma to the sequence (A_δ, ψ_δ) to conclude that there exists $\psi \in L^{1,2}(\mathbb{S}_L)$ such that $\|\psi\| = 1$ and

$$Z_{A,\infty}\psi = T_A\psi = 0$$

where $A = \lim A_\delta$. The discussion preceding Corollary 3.8 shows that this thing is impossible when $c_1(L) \in \mathcal{V}_{g,\ell}$. The proposition is proved. $\square$

4 Reversing the adiabatic process

In this final section we describe a range of spin^c structures on a circle bundle over a surface for which $\mathcal{A}_\infty = \lim_{\delta\to\infty} \mathcal{S}_\delta$. This boils down to constructing irreducible solutions of the SW_δ equations starting from irreducible adiabatic solutions.

§4.1 Statement of the main result We continue to use the same notations as in §3.3. Fix a line bundle $L_\Sigma \to \Sigma$ and set $L = \pi^* L_\Sigma \to N$.

For $|\ell| \geq 2g - 1 \geq 0$ define

$$\mathcal{R}_{\ell,g} = \{0\} \cup \{g-1, \cdots, |\ell| - g + 1\} \subset \mathbb{Z}_\ell.$$

The main result of this section is the following.

Theorem 4.1 If $|\ell| \geq 2g-1$ and $c_1(L) \in \mathcal{R}_{\ell,g}$ then

$$\lim_{\delta\to\infty} \mathcal{S}_\delta(g,\ell,L) = \mathcal{A}(g,\ell,L)$$

i.e. for any $(\phi, A) \in \mathcal{A}_\infty$ there exists a solution (ϕ_δ, A_δ) of SW_δ which (modulo $\mathfrak{G}_L$ converges in $L^{1,2}$-norm to (ϕ, A).

Note that when $|\ell| \geq 2g$ and $c_1(L) \neq \pm(g-1) \pmod{\ell}$ this result is clear since in this case $\mathcal{A}_\infty$ consists only of reducible solutions so that trivially

$$\mathcal{A}_\infty \subset \mathcal{S}_\delta, \ \ \forall\delta.$$

Thus we have to consider only the case $|\ell| \geq 2g-1$ and $c_1(L) = \pm(g-1) \bmod \ell$. Under this assumption the moduli space of adiabatic solutions consists of two components. A torus of reducible solutions and a unique irreducible solution (A_0, ϕ_0). The reducible solutions are obviously adiabatic limits so we only have to prove that (A_0, ϕ_0) can be approximated by solutions of SW_δ, $\delta \gg 1$.

We will consider only the case $c_1(L) = -(g-1)$. The assumption $|\ell| \geq 2g-1$ shows the *unique* line bundle $L_\Sigma \to \Sigma$ such that $\pi^* L_\Sigma = L$ is $K_\Sigma^{-1/2}$. Thus $L = \mathcal{K}^{-1/2}$. In this case

$$\mathbb{S}_L \cong \mathcal{K}^{-1} \otimes \underline{\mathbb{C}}$$

This unique (modulo $\mathfrak{G}_L$) irreducible adiabatic solution is (ϕ_0, A_0) where A_0 is the pullback of the Levi-Civita connection on $K_\Sigma^{-1/2}$ while $\phi = 0 \oplus c_0$ where c_0 is a complex constant normalized by

$$\frac{1}{2}|c_0|^2 = -\mathbf{i} F_{A_0}(\zeta_1, \zeta_2). \tag{4.23}$$

Modulo a constant gauge transformation we may assume c_0 is a *real, positive* constant.

Before we begin the proof of Theorem 4.1 we want to make a few elementary observations.

For any connection A on L define

$$\mathcal{D}_{A,\infty} = Z_{A,\infty} + T_A.$$

When A is the connection A_0 discussed above we will write simply $\mathcal{D}_\infty$. The equality (1.13) implies

$$\mathcal{D}_{A,\delta} = \delta Z_{A,\infty} + T_A + \lambda_\delta/2.$$

In particular

$$\mathcal{D}_\delta \phi_0 = \lambda_\delta/2\phi_0 \tag{4.24}$$

where $\mathcal{D}_\delta := \mathcal{D}_{A_0,\delta}$.

Note also that if $\psi = \psi_0 \oplus \psi_1 \in \ker \mathcal{D}_\infty$ is such that $Z_{A_0,\infty}\psi = T_{A_0}\psi = 0$, $\psi_0 \equiv 0$ and

$$\int_N \mathfrak{Re}\langle \psi, \mathbf{i}\phi_0 \rangle \, dv_1 = 0$$

then ψ must be a *real* multiple of ϕ_0. This simple observation will play a crucial role in estimating the eigenvalues of the linearizations of SW_δ.

§4.2 The set-up An important part of the proof of Theorem 4.1 is the construction of a suitable functional framework. This is what will be accomplished in this subsection.

We fix A_0 as a reference connection. Any other connection on L will have the form $A = A_0 + a$, $a \in \mathbf{i}\Omega^1(N)$. Set for simplicity

$$F_0 = F_{A_0},\ \ F_a = F_{A_0+a},\ \ ,\mathcal{D}_\delta = \mathcal{D}_{A_0,\delta},\ \ \mathcal{D}_a^\delta = \mathcal{D}_{A_0+a,\delta}.$$

The configuration space for the SW-equations is

$$\mathcal{C} = L^{1,2}(\mathbb{S}_L \oplus \mathbf{i}T^*N).$$

The elements of $\mathcal{C}$ will be generically denoted by $\mathfrak{c} = (\psi, a)$. This configuration space is acted upon by the gauge group $\mathfrak{G}_L$ consisting of $L^{2,2}$-gauge transformations of L. As usual, a δ-subscript will indicate the corresponding object is defined in terms of the deformed metric g_δ. The solutions of SW_δ are critical points of the functional

$$\mathfrak{f}_\delta : \mathcal{C}_\delta \to \mathbb{R},\ \ \mathfrak{f}_\delta(\psi, a) = \frac{1}{2}\int_N a \wedge (F_0 + F_a) + \frac{1}{2}\int_N \langle \psi, \mathcal{D}_a^\delta \psi\rangle dv_\delta.$$

Lemma 3.1 shows that the L^2-gradient of $\mathfrak{f}_\delta$ is

$$\nabla \mathfrak{f}_\delta\,|_{(\psi,a)} = \mathcal{D}_a^\delta\psi \oplus \mathbf{c}_\delta^{-1}(\tau_\delta(\psi)) - *F_a).$$

Define

$$\mathcal{W}_\delta = \{a \in L^{1,2}(\mathbf{i}T^*N)\ ;\ d_\delta^* a = 0\}$$

and denote by $\bar{\mathcal{W}}_\delta$ its L^2-closure. Hodge theory shows that any orbit of $\mathfrak{G}_L$ in $\mathcal{C}$ intersects $\mathcal{B}_\delta = L^{1,2}(\mathbb{S}_L) \oplus \mathcal{W}_\delta$ along a discrete set so that in detecting the critical points of $\mathfrak{f}_\delta$ it suffices to study the restriction of $\mathfrak{f}_\delta$ to $\mathcal{B}_\delta$. Since the orbits of $\mathfrak{G}_L$ do not intersect $\mathcal{B}_\delta$ orthogonally the L^2-gradient of the restriction to $\mathcal{B}_\delta$ is

$$\nabla \mathfrak{f}_\delta\,|_{(\psi,a)} = \mathcal{D}_a^\delta\psi \oplus P_\delta(\tau_\delta(\psi) - *_\delta F_a)$$

where P_δ denotes the L^2_δ-orthogonal projection onto $\bar{\mathcal{W}}_\delta$. The new configuration space $\mathcal{B}_\delta$ has a residual S^1-action- the constant gauge transformations. Thus if $(\psi, a) \in \mathcal{B}_\delta$ then

$$\exp(\mathbf{i}\theta)(\psi, a) = \exp(\mathbf{i}\theta)\cdot\psi, a).$$

The fixed point set of this action is $0 \oplus \mathcal{W}_\delta$. Set

$$\mathcal{B}_\delta^* = \{(\psi, a) \in \mathcal{B}_\delta\ ;\ \psi \neq 0\}.$$

The tangent space to the S^1-orbit through $\mathfrak{c} = (\psi, a)$ is $\mathcal{O}_\mathfrak{c} = \mathbb{R}(\mathbf{i}\psi, 0)$. The L^2-orthogonal projection onto $\mathcal{O}_\mathfrak{c}^\perp$ is

$$Q_\mathfrak{c}(\dot\psi, \dot a) = (\psi - \|\psi\|_\delta^{-1}\mathfrak{Re}\left(\int_N \langle \dot\psi, \mathbf{i}\psi\rangle dv_\delta\right)\mathbf{i}\psi\ ,\ \dot a).$$

We will work on the quotient $\mathcal{B}^*_\delta/S^1$. This is a smooth Banach manifold. The L^2-gradient of the functional induced by $\mathfrak{f}_\delta$ on this quotient is

$$\nabla \mathfrak{f}_\delta|_{\mathfrak{c}} = Q_{\mathfrak{c}} \mathfrak{D}^\delta_a \psi \oplus P_\delta(\tau_\delta(\psi) - *_\delta F_a).$$

To compute the linearization of this gradient we will work "upstairs" and then orthogonally project "downstairs".

On $\mathcal{C}_\delta$ the linearization of the gradient at $\mathfrak{c} = (\psi, a)$ is

$$\mathfrak{F}_{\mathfrak{c}}(\dot{\psi}, \dot{a}) = (\mathfrak{D}^\delta_a + \mathbf{c}_\delta(\dot{a})\psi) \oplus (\dot{\tau}_\delta(\psi, \dot{\psi}) - *_\delta d\dot{a})$$

where $\dot{\tau}_\delta(\psi, \cdot)$ denotes the linearization of τ_δ at ψ and it is given by (dropping the δ subscript for simplicity)

$$\dot{\tau}(\psi, \dot{\psi}) = \frac{1}{2}\sum_{i=0} 2\left(\langle \dot{\psi}, \mathbf{c}(\eta_i)\psi\rangle + \langle \psi, \mathbf{c}(\eta_i)\dot{\psi}\rangle\right)\eta_i$$

$$= \mathbf{i}\sum_{i=0}^{2} \mathfrak{Im}\langle \psi, \mathbf{c}(\eta_i)\dot{\psi}\rangle \eta_i.$$

When ψ is the spinor $\phi_0 = (0, c_0)$ defined in the previous subsection we set $\dot{\tau}_\delta(\dot{\psi}) := \dot{\tau}_\delta(\phi_0, \dot{\psi})$. If we write $\dot{\psi}$ as $\dot{\psi}_0 \oplus \dot{\psi}_1 \in L^{1,2}(\mathcal{K}^{-1} \oplus \underline{\mathbb{C}}$ then a simple computation shows that, viewed as an endomorphism of $\mathbb{S}_L$, $\dot{\tau}(\dot{\psi})$ is given by

$$\dot{\tau}(\dot{\psi}) = c_0 \begin{bmatrix} -\mathfrak{Re}\dot{\psi}_1 & \dot{\psi}_0 \\ \bar{\dot{\psi}}_0 & \mathfrak{Re}\dot{\psi}_1 \end{bmatrix}.$$

The linearization of $\nabla \mathfrak{f}_\delta$ at $[\mathfrak{c}] \in \mathcal{B}_\delta/S^1$ is

$$\mathfrak{L}^\delta_{\mathfrak{c}} \begin{bmatrix} \dot{\psi} \\ \dot{a} \end{bmatrix} = \begin{bmatrix} Q_{\mathfrak{c}}(\mathfrak{D}^\delta_a \dot{\psi} + \mathbf{c}_\delta(\dot{a})\psi) \\ P_\delta(\dot{\tau}_\delta(\psi, \dot{\psi}) - *_\delta d\dot{a}) \end{bmatrix}$$

$\forall(\dot{\psi}, \dot{a})$ such that

$$\mathfrak{Re}\int_N \langle \dot{\psi}, \mathbf{i}\psi\rangle dv_g = 0, \quad d^*_\delta \dot{a} = 0.$$

Denote by $\mathfrak{L}^\delta_0$ the linearization at $\mathfrak{c}_0 = (\phi_0, A_0)$ and set

$$X_\delta = \mathcal{O}^\perp_{\mathfrak{c}_0}$$

equipped with the L^2_δ-norm. We can regard $\mathfrak{L}^\delta_0$ as a densely defined unbounded operator $X_\delta \to X_\delta$. Its domain is

$$Y_\delta = X_\delta \cap L^{1,2}_\delta. \tag{4.25}$$

Our next result will play a crucial role in the proof of Theorem 4.1

Proposition 4.2 (a) The unbounded operator $\mathfrak{L}_\delta$ is closed, selfadjoint and Fredholm.
(b) Set $z_\delta = \mathrm{dist}(0, \sigma(\mathfrak{L}_\delta))$ where $\sigma(\mathfrak{L}_0^\delta)$ denotes the spectrum of $\mathfrak{L}_0^\delta$. Then

$$z_\delta \sim \delta^{-1}$$

where for any two functions $f(\delta)$ and $g(\delta)$ we set

$$f \sim g \Longleftrightarrow \exists c > 1;\; c^{-1} f(\delta) \leq g(\delta) \leq c f(\delta), \;\; \forall \delta \gg 1.$$

Proof The first part follows easily using the ellipticity of $\mathcal{D}^\delta$ and of the Hodge operator $d + d_\delta^*$. The details are left to the reader.

To prove the second part we argue by contradiction. Note first that the equality $\mathcal{D}_\delta \phi_0 = \frac{\lambda}{2\delta}\phi_0$ implies that $z_\delta = O(\delta^{-1})$. Assume now that for all $\delta \gg 1$ there exists a pair $(\dot{\psi}_\delta, \dot{a}\delta) \in Y_\delta$ and $\mu_\delta \in \mathbb{R}$ such that

$$\left\{ \begin{array}{rcl} Q_\delta(\mathcal{D}_\delta \dot{\psi} + \mathbf{c}_\delta(\dot{a})\psi) & = & \mu_\delta \dot{\psi}_\delta \\ P_\delta \mathbf{c}_\delta^{-1}(\dot{\tau}_\delta(\dot{\psi})) - *_\delta d\dot{a} & = & \mu_\delta \dot{a}_\delta \end{array} \right. \tag{4.26}$$

$$\|\dot{\psi}_\delta\| + \delta^{1/2}\|\dot{a}_\delta\|_\delta = 1, \;\; |\mu_\delta| = z_\delta = o(\delta^{-1}). \tag{4.27}$$

(Recall that $\|\cdot\|_\delta$ denotes the L^2-norm defined in terms of g_δ.) Write as usual $\dot{\psi}_\delta = \dot{\psi}_0^\delta \oplus \dot{\psi}_1^\delta$. Since

$$\mathfrak{Re} \int_N \langle \dot{\psi}_\delta, \mathbf{i}\phi_0 \rangle dv_\delta = 0$$

we deduce $\mathfrak{Im}\dot{\psi}_1^\delta = 0$. On the other hand

$$\int_N \mathfrak{Re}\langle \mathbf{c}_\delta(\dot{a}_\delta)\phi_0, \mathbf{i}\phi_0 \rangle dv_\delta = 0.$$

Moreover, since $\mathcal{D}_\delta = \delta Z_{A_0,\infty} + T_{A_0} + \lambda_\delta/2$ we deduce that if $\dot{\psi} \perp \mathbf{i}\phi_0$ then

$$\int_N \mathfrak{Re}\langle \mathcal{D}_\delta \dot{\psi}, \phi_0 \rangle dv_\delta = \frac{\lambda_\delta}{2} \int_N \mathfrak{Re}\langle \dot{\psi}, \mathbf{i}\phi_0 \rangle dv_\delta = 0$$

The above two equalities show that we can drop the Q_δ term from the first equation in (4.26).

Lemma 4.3 $\dot{\psi}_\delta \neq 0$ for all $\delta \gg 1$.

Proof Assume the contrary i.e. $\dot{\psi}_\delta \equiv 0$ for a subsequence $\delta \to \infty$. Locally we can write

$$\dot{a}_\delta = \mathbf{i}u_\delta \eta_\delta + v_\delta \varepsilon - \bar{v}_\delta \bar{\varepsilon}$$

where u_δ is real while v_δ is complex valued. We have

$$\mathbf{c}_\delta(\dot{a}_\delta)\phi_0 = c_0 \left[\begin{array}{c} \bar{v}_\delta \bar{\varepsilon} \\ u_\delta \end{array} \right]. \tag{4.28}$$

The the first equation of the system (4.26) becomes

$$\mathbf{c}_\delta(\dot{a}_\delta)\phi_0 = 0$$

which implies $\dot{a}_\delta \equiv 0$. This contradicts (4.27). $\Box$

The decisive moment in the proof of Proposition 4.2 is contained in the following result.

Lemma 4.4 Set $\Psi_\delta = \|\dot{\psi}_\delta\|^{-1}\dot{\psi}_\delta$. Then there exists $t \in \mathbb{R} \setminus \{0\}$ and a subsequence $\delta \to \infty$ such that $\Psi_\delta \to t\phi_0$ in $L^{1,2}$.

The proof of Lemma 4.4 relies on the following auxiliary result which is an immediate consequence of Proposition 3.45 of [BGV].

Lemma 4.5 For any $\dot{a} \in \mathcal{W}_\delta$

$$\mathfrak{D}_\delta \mathbf{c}_\delta(\dot{a}) + \mathbf{c}_\delta(\dot{a})\mathfrak{D}_\delta = -2\nabla^\delta_{\dot{a}} + \mathbf{c}_\delta(*_\delta d\dot{a})$$

where ∇^δ denotes the spin^c connection defined by A_0 on $\mathbb{S}_L$ using the metric g_δ while $\nabla^\delta_{\dot{a}}$ denotes the covariant derivative along the vector field g_δ-dual to $\dot{a}$. (Note that $\dot{a}$ is by definition a purely imaginary form and so will be its dual vector field.)

Proof of Lemma 4.4 Consider the complex valued 1-form $\omega_\delta = \mathbf{i}\mathfrak{Re}\dot{\psi}_1^\delta \eta_\delta + \bar{\dot{\psi}}_0^\delta - \dot{\psi}_0^\delta$. Then $c_0 \mathbf{c}_\delta(\omega_\delta) = \dot{\tau}_\delta(\dot{\psi}_\delta)$ so that

$$\int_N \langle \mathbf{c}_\delta P_\delta \mathbf{c}_\delta^{-1} \left(\dot{\tau}_\delta(\dot{\psi}_\delta) \right) \, \phi_0, \dot{\psi}_\delta \rangle dv_\delta$$

$$= \int_N \langle P_\delta \mathbf{c}_\delta^{-1}(\dot{\tau}_\delta(\dot{\psi}_\delta)) \, , \, \omega_\delta \rangle dv_\delta = c_0 \int_N \langle P_\delta \omega_\delta, \omega_\delta \rangle dv_\delta \geq 0.$$

Using the second equation in (4.26)

$$\mathfrak{Re} \int_N \langle \mathbf{c}_\delta(*_\delta d\dot{a}_\delta)\phi_0, \dot{\psi}_\delta \rangle dv_\delta \geq -\mathfrak{Re}\mu_\delta \int_N \langle \mathbf{c}_\delta(\dot{a}_\delta)\phi_0, \dot{\psi}_\delta \rangle dv_\delta. \tag{4.29}$$

Applying $\mathfrak{D}_\delta$ to the first equation in (4.26) equation (remember Q_δ need not be included) we deduce using Lemma 4.5

$$\mathfrak{D}_\delta^2 \dot{\psi}_\delta - \mathbf{c}_\delta(\dot{a}_\delta)\mathfrak{D}_\delta \phi_0 - 2\nabla^\delta_{\dot{a}_\delta}\phi_0 + \mathbf{c}(*_\delta d\dot{a}_\delta)\phi_0 = \mu_\delta \mathfrak{D}_\delta \dot{\psi}_\delta. \tag{4.30}$$

This equality can be further simplified using the equalities

$$\mathcal{D}_\delta \phi_0 = \frac{\lambda}{2\delta}\phi_0$$

and since ∇^δ is the tensor of the spin connection on $\mathcal{K}^{-1/2} \oplus \mathcal{K}^{1/2}$ with the pullback connection A_0 on $\mathcal{K}^{-1/2} = \pi^* K_\Sigma^{-1/2}$ we deduce from the computations in §2.4 that

$$\nabla_{\dot{a}_\delta}\phi_0 = c_0 \frac{\lambda}{2\delta}((1+\mathbf{i})\bar{v}_\delta \oplus \mathbf{i} u_\delta) \tag{4.31}$$

Taking the (*real*) inner product of (4.30) with $\dot{\psi}_\delta$ and integrating by parts we deduce

$$\|\mathcal{D}_\delta \dot{\psi}_\delta\|_\delta^2 - 2\mathfrak{Re}\langle \nabla_{\dot{a}_\delta}\phi_0, \dot{\psi}_\delta\rangle_\delta + \mathfrak{Re}\langle \mathbf{c}_\delta(*d\dot{a}_\delta)\phi_0, \dot{\psi}_\delta\rangle_\delta - \frac{\lambda}{2\delta}\mathfrak{Re}\langle \mathbf{c}_\delta(\dot{a}_\delta)\phi_0, \dot{\psi}_\delta\rangle_\delta$$

$$= \mu_\delta \langle \mathcal{D}_\delta \dot{\psi}_\delta, \dot{\psi}_\delta\rangle_\delta$$

Using the inequality (4.29) and the identity (4.31) we deduce the following inequality (c will generically denote a constant independent of δ).

$$\|\mathcal{D}_\delta \dot{\psi}_\delta\|_\delta^2 - \frac{c}{\delta}\|\dot{a}_\delta\|_\delta \cdot \|\dot{\psi}_\delta\|_\delta - |\mu_\delta| \cdot \|\dot{a}_\delta\|_\delta \cdot \|\dot{\psi}_\delta\|_\delta \leq |\mu_\delta| \cdot \|\mathcal{D}_\delta \dot{\psi}_\delta\|_\delta \cdot \|\dot{\psi}_\delta\|_\delta.$$

This inequality implies

$$\|\mathcal{D}_\delta \dot{\psi}_\delta\|_\delta = O(\delta^{-1}\|\dot{\psi}\|_\delta)$$

which is equivalent to

$$\|\mathcal{D}_\delta \Psi_\delta\| = O(\delta^{-1}), \ \ (\|\Psi_\delta\| = 1).$$

Using the adiabatic decoupling lemma (or rather the remark following its proof) we deduce that on a subsequence Ψ_δ converges in $L^{1,2}$ to a *nontrivial* element $\Psi_\infty \in \ker Z_{A_0,\infty} \cap \ker T_{A_0}$ such that $\mathfrak{Re}\langle \Psi_\infty, \mathbf{i}\phi_0\rangle = 0$. Lemma 4.4 now follows using the observation at the end of §4.1. □

We can now complete the proof of the Proposition 4.2. Take the inner product with ϕ_0 of the first equation in (4.26) to deduce

$$\left(\frac{\lambda}{2\delta} - \mu_\delta\right)\int_N \langle \dot{\psi}_\delta, \phi_0\rangle dv_1 = -\int_N \langle \mathbf{c}_\delta(\dot{a}_\delta)\phi_0, \dot{\psi}_\delta\rangle dv_1$$

so that

$$\left(\frac{\lambda}{2\delta} - \mu_\delta\right)\int_N \langle \Psi_\delta, \phi_0\rangle dv_1 = -\int_N \langle \mathbf{c}(\dot{a}_\delta)\phi_0, \Psi_\delta\rangle dv_1. \tag{4.32}$$

By taking the inner product with $\dot{a}_\delta \in \mathcal{W}_\delta$ of the second equation in (4.26) we get after integrating by parts and using (4.27)

$$\left|\int_N \langle P_\delta \mathbf{c}_\delta^{-1}(\dot{\tau}_\delta(\dot{\psi}_\delta)), \dot{a}_\delta\rangle_\delta dv_1\right| = \left|\int_N \langle \mathbf{c}_\delta^{-1}(\dot{\tau}_\delta(\dot{\psi}_\delta)), \dot{a}_\delta\rangle_\delta dv_1\right| = O(|\mu_\delta|\delta\|\dot{a}_\delta\|_\delta^2) = O(|\mu_\delta|).$$

A simple computation shows that

$$\left|\int_N \langle \mathbf{c}_\delta^{-1}(\dot\tau_\delta(\dot\psi_\delta)), \dot a_\delta\rangle_\delta dv_1\right| = \left|\int_N \langle \mathbf{c}(\dot a_\delta)\phi_0, \dot\psi_\delta\rangle dv_1\right|$$

Hence

$$\left|\int_N \langle \mathbf{c}(\dot a_\delta)\phi_0, \Psi_\delta\rangle dv_1\right| = O(|\mu_\delta|) = o(\delta^{-1})$$

Using this last estimate in (4.32) we deduce

$$\left(\frac{\lambda}{2} - \delta\mu_\delta\right)\int_N \langle \Psi_\delta, \phi_0\rangle dv_1 = o(1)$$

We now let $\delta \to \infty$ in the above equality. Lemma 4.4 leads to a contradiction. The proposition is proved. □

§4.3 Proof of the main result As was indicated at the beginning of §4.1 we have to show that for all $\delta \gg 1$ there exists a solution (ϕ_δ, A_δ) of SW_δ which converges (modulo $\mathfrak{G}_L$) to (ϕ_0, A_0) as $\delta \to \infty$. To produce such solutions we will use the technique pioneered by Taubes in [T] where he proved the existence of self-dual connections. The abstract result behind this technique is contained in the following version of the inverse function theorem. We will state it in the simplest context of maps between two Banach spaces. The extension to Banach manifolds and vector bundles is only notationally more complicated.

Lemma 4.6 Consider a map between two Banach spaces $F : Y \to X$ which can be decomposed as $F(y) = Ly + N(y)$ where L and N satisfy the following conditions.
(i) L is a linear isomorphism of Banach spaces. Set

$$\mu = \inf_{y\neq 0} \frac{\|Ly\|}{\|y\|}$$

(ii) There exists $\kappa > 0$ such that

$$\|N(y_1) - N(y_2)\| \leq \kappa(\|y_1\| + \|y_2\|)(\|y_1 - y_2\|), \ \ \forall y_1, y_2 \in Y.$$

Set $\varepsilon_0 = \|N(0)\|$ and $q = \kappa\mu^{-1}$. Suppose that

$$\varepsilon_0 < \frac{1}{4q} < \frac{1}{2}. \tag{4.33}$$

and denote by $r = r(q, \varepsilon_0)$ the smallest root of the quadratic equation

$$qr^2 - r + \varepsilon_0 = 0. \tag{4.34}$$

Then there exists an unique $y_0 \in Y$ such that

$$\|y_0\| \leq r(q, \varepsilon_0) \ \text{ and } \ F(y_0) = 0.$$

The idea of this lemma is intuitively clear. If 0 is an "almost" solution of the equation $F(y) = 0$ (ε_0 quantifies the attribute "almost") and the linearization L is invertible then we can perturb 0 to a genuine solution provided the norm of the inverse is sufficiently small (the attribute "sufficiently small" is clarified in the inequality (4.31) which also incorporates an interaction with the nonlinear term).

The proof is an immediate application of the Banach fixed point theorem and can be safely left to the reader.

In our case the equation we want to solve is

$$\nabla \mathfrak{f}_\delta = 0$$

near $\mathfrak{c}_0$ and $\nabla \mathfrak{f}_\delta$ is rather a section of an infinite dimensional vector bundle. Trivializing (via parallel transport for example) near this configuration we can regard it as an equation of the form

$$\mathcal{F}_\delta(y) = 0, \quad y \in Y_\delta$$

where X_δ and Y_δ are defined in (4.25) and $\mathcal{F} : Y_\delta \to X_\delta$. The role of the operator L in Lemma 4.6 is played by the linearization $\mathfrak{L}_0^\delta$ this time viewed as a *bounded* operator $Y_\delta \to X_\delta$. As usual c will generically denote a constant independent of $\delta \gg 1$. The first thing we want to prove is the following.

Lemma 4.7 There exists $c > 0$ such that

$$\|\mathfrak{L}_0^\delta(\psi, a)\|_{2,\delta} \geq \frac{c}{\delta}(\|\psi\|_{1,2,\delta} + \|a\|_{1,2,\delta}), \quad \forall (\psi, a) \in Y_\delta.$$

Proof Note that there exists $c > 0$ such that

$$\|\mathfrak{L}_0^\delta(\psi, a)\|_{2,\delta}^2 + \|\psi\|_{2,\delta}^2 + \|a\|_{2,\delta}^2 \geq c\left(\|\mathcal{D}_\delta\psi\|_{2,\delta}^2 + \|da\|_{2,\delta}^2\right).$$

The Weitzenböck formula for $\mathcal{D}_\delta$ coupled with the boundedness of F_{A_0} and of the scalar curvature of N_δ show that

$$\|\psi\|_{1,2,\delta} \leq c\left(\|\mathcal{D}_\delta\psi\|_{2,\delta} + \|\psi\|_{2,\delta}\right).$$

Similarly, the Bochner formula for the Hodge operator on 1-forms coupled with the boundedness of the Ricci curvature shows that there exists $c > 0$ such that

$$\|a\|_{1,2,\delta} \leq c\left(\|da\|_{2,\delta} + \|a\|_{2,\delta}\right), \quad \forall a \in \mathcal{W}_\delta.$$

Putting all the above together we deduce there exists $c > 0$ such that

$$\|\psi\|_{1,2,\delta} + \|a\|_{1,2,\delta} \leq c\left(\|\mathfrak{L}_0^\delta(\psi, a)\|_{2,\delta} + \|a\|_{2,\delta} + \|\psi\|_{2,\delta}\|\right).$$

The lemma now follows from Proposition 4.2. $\square$

The nonlinear part $\mathcal{N}_\delta$ in the Seiberg-Witten equation comes only from the quadratic term τ_δ. Thus in this case we have (via the Hölder inequalities)

$$\|\mathcal{N}(\psi_1, a_1) - \mathcal{N}_\delta(\psi_2, a_2)\|_{2,\delta} \leq c(\|\psi_1\|_{4,\delta} + \|\psi_2\|_{4,\delta}) \cdot \|\psi_1 - \psi_2\|_{4,\delta}.$$

At this point the controlled geometry of the deformation $g \to g_\delta$ plays a magical role. More precisely, since the Ricci curvature is bounded, $\mathrm{vol}\,(N_\delta) \sim \delta^{-1}$ and $\mathrm{diam}\,(N_\delta) \sim 1$ we deduce from the Sobolev estimates of [Be], Appendix VI that there exists $c > 0$ such that for any $f \in L^{1,2}_\delta(N_\delta)$ we have the *sharp* Sobolev inequality

$$\|f\|_{4,\delta} \leq c\delta^{-1/4}(\|df\|_{2,\delta} + \|f\|_{2,\delta})$$

Kato's inequality now implies immediately that for every $\psi \in L^{1,2}_\delta(\mathbb{S}_L)$ we have

$$\|\psi\|_{4,\delta} \leq c\delta^{-1/4}\|\psi\|_{1,2,\delta}.$$

Putting all the above together we deduce

$$\|\mathcal{N}(\psi_1, a_1) - \mathcal{N}_\delta(\psi_2, a_2)\|_{2,\delta} \leq c\delta^{-1/2}(\|\psi_1\|_{1,2,\delta} + \|\psi_2\|_{1,2,\delta})(\|\psi_1 - \psi_2\|_{1,2,\delta}).$$

Finally, using (4.23) and (4.24) we deduce $\|\nabla \mathfrak{f}_\delta(\mathfrak{c}_0)\|_{2,\delta} \sim \delta^{-3/2}$.

We now want to apply Lemma 4.6 with

$$\varepsilon_0 \sim \delta^{-3/2}, \;\; \mu = O(\delta^{-1}) \;\text{ and }\; \kappa = O(\delta^{-1/2}).$$

The only problem is that the solution it postulates might be a reducible one. We need to eliminate this possibility. This is where sharp asymptotics of the solutions of (4.34) are needed. The quadratic formula shows

$$r(q, \varepsilon_0) = \frac{1 - \sqrt{1 - 4q\varepsilon_0}}{4q}$$

where $\varepsilon_0(\delta) \sim \delta^{-3/2}$. Thus

$$r(q, \varepsilon_0) \sim \varepsilon_0(\delta) \sim \delta^{-3/2}.$$

On the other hand the distance from ϕ_0 to the reducible set $0 \oplus \mathcal{W}_\delta$ is $\|\phi_0\|_{2,\delta} \sim \delta^{-1/2}$. Hence for $\delta \gg 1$ the solution detected with the help of Lemma 4.6 is too close to (ϕ_0, A_0) to be reducible. Theorem 4.1 is proved. $\Box$

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